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Optimal Mismatched Filter Design for Radar Ranging, Detection and Resolution

R. J. McAulay
J. R. Johnson

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R. J. McAulay

Group 41

J. R. Johnson

Group 32

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ABSTRACT

In a multiple target environment a radar signal processor often uses weighting filters which are not necessarily matched to the transmitted waveform. In this paper expressions for the mean-square range-estimation error, the detection signal-to-noise (SNR) and the effects of sidelobes are derived in terms of the impulse response of an arbitrary mismatched filter. It is desired to find that impulse response which results in the minimum range estimate variance subject to preassigned constraints on the sidelobes and the detection SNR. This optimization problem is first formulated in state-space in which the optimal control law is sought. Pontryagin's maximum principle is used to obtain necessary conditions for the optimum impulse response, from which it is possible to deduce the structure of the optimum filter. Certain mathematical details which detract from the rigor of the time domain formulation are resolved by formulating the problem in the frequency domain and applying Hilbert space techniques. It is shown that for the problem of detecting the radar target and estimating its range, the optimum filter is a modified transversal equalizer. If only the detection function is to be performed the optimum filter reduces to the transversal equalizer. This establishes the optimality of this important practical device as the solution to the radar detection problem in a multiple target environment. The tap weights and spaces of the delay line as well as certain other parameters upon which the solution depends can be found by solving a non-linear programming problem. Numerical results are given for an interesting class of transmitted waveforms which shows the tradeoffs of the various filter parameters.

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Chief, Lincoln Laboratory Office

I. Introduction

It has long been known that the matched filter represents the "optimum" means of processing data for obtaining estimates of target range. Optimality in this case means that the estimates have the smallest mean-square error possible, a result which is valid only when the signal-to-noise ratio is large. In many applications target resolvability is a consideration almost as important as range accuracy, and it is then not clear that the matched filter is the best receiver to use. In fact, for some waveforms, a sinusoidal pulse, for example, very good range accuracy can be obtained, but the resolution is poor because the envelope of the matched filter output signal has large subsidiary sidelobes.

One approach to this problem is to assume that the receiver is a matched filter and then try to design the input signal that will produce good range estimates subject to constraints on the sidelobe structure of the compressed pulse. Algorithms are now available to generate the solution to this problem [1], but in most cases the resulting waveform is quite complicated, and it is difficult to build the matched filter for it.

The other approach is to use a known signal which can be transmitted easily, and to use a mismatched filter. This has been done for the linear FM waveform [2] in an effort to reduce the sidelobes of the compressed pulse, but at the expense of a loss in signal detectability and range accuracy. Heretofore, no effort has been made to design a mismatched filter to simultaneously minimize the mean-square range error and loss of signal detectability subject to pre-assigned constraints on the sidelobe structure.

In this paper we assume that a given pulse is received in the presence of additive white Gaussian noise and passed through a filter which is not necessarily matched to the input pulse. We assume that the range is estimated by locating the time at which the envelope of the filter output is maximum. The performance of this estimation scheme has been analysed previously [3] with respect to measuring the loss of accuracy and detectability due to non-optimum filtering. It is shown that one's inability to build perfectly matched filters does result in a loss in performance. However, the advantages of mismatching with respect to improving the multiple target resolution is not discussed.

In Section II we give a brief derivation of the equations for the mean-square range estimation error in terms of an arbitrary filter impulse response. The effects of multiple targets are discussed and design constraints which provide for good resolution are developed. Since the processor is no longer matched to the transmitted signal a degradation in detection signal-to-noise must be expected. (The detection signal-to-noise ratio for the mismatched filter is derived). Therefore, we define the cost function as a weighted combination of the range estimate variance and the detection noise-to-signal ratio. This quantity is to be minimized subject to the sidelobe constraints. The weighting in the cost function can be varied to emphasize either the detection performance or the estimation performance of the filter. This weighting factor shows up as a key parameter in the structure of the optimal filter. We can then associate the detection and estimation functions of the filter with its various substructures.

The filter design problem is then formulated as an optimal control problem in state space. By following the recipe described by Pontryagin's maximum

principle we can find the structure of the optimal filter. Certain mathematical details concerned with the stability of the resulting structure on an infinite time interval detract from the usefulness of the state-space technique. By formulating the problem in the frequency domain and using the insights obtained from the optimal control solution, a mathematically rigorous derivation of the optimum filter is obtained using the projection theorem in a suitably defined Hilbert Space. The optimum filter is shown to be a modified transversal equalizer. If the weighting factor were chosen to weight only the detection signal-to-noise ratio in the original cost functional, the optimum filter reduces to the transversal equalizer. Therefore, the optimality of this structure for target detection is established rigorously for the first time in the published literature. For weighting factors which include some measure of the estimation performance it is further established that the transversal equalizer is not the optimal processor but it is an integral part of the optimum filter structure.

Certain constants remain to be determined to completely specify the optimum filter. For example, the tap gains and spaces of the transversal equalizer have to be determined for a particular sidelobe constraint function. It is shown that these constraints can be solved using nonlinear programming techniques. To show the feasibility of the method and the tradeoffs of the various filtering parameters we conclude the paper with a numerical evaluation of the filter for an interesting class of transmitted waveforms.

II The Suboptimal Signal Processor

In this section we shall derive the performance criteria to be used in the filter design problem. We shall first concern ourselves with the variance of the range estimate when a mis-matched filter is used in the usual matched filter or maximum-likelihood-estimator configuration. The derivation closely follows that in reference [3] but is included here to make clear the essential assumptions as these become important when the criterion is to be used for filter synthesis. The system analysed is shown in Figure 1. The input consists of a bandpass signal $S(t)$ plus white stationary bandpass Gaussian noise $N(t)$. The filter, with the transfer function $Q(\omega) = H(\omega-\omega_c) + H(\omega+\omega_c)$, is square-law detected. The time-delay corresponding to the estimate of radar range is then taken as the time at which this envelope assumes its maximum value. If the filter were matched to the signal, then this processor is precisely the maximum likelihood estimator and has certain optimal properties. Since the filter is not matched to the signal the processor is merely one specific suboptimal method of estimating radar range, (it is widely used in practice).

The complex envelope of the input signal is

$$r(t) = Ae^{j\theta} p(t-\tau_o) + n(t) \tag{1}$$

where

A = the signal amplitude

$p(t)$ = the complex signal envelope

θ = the random phase of the received signal

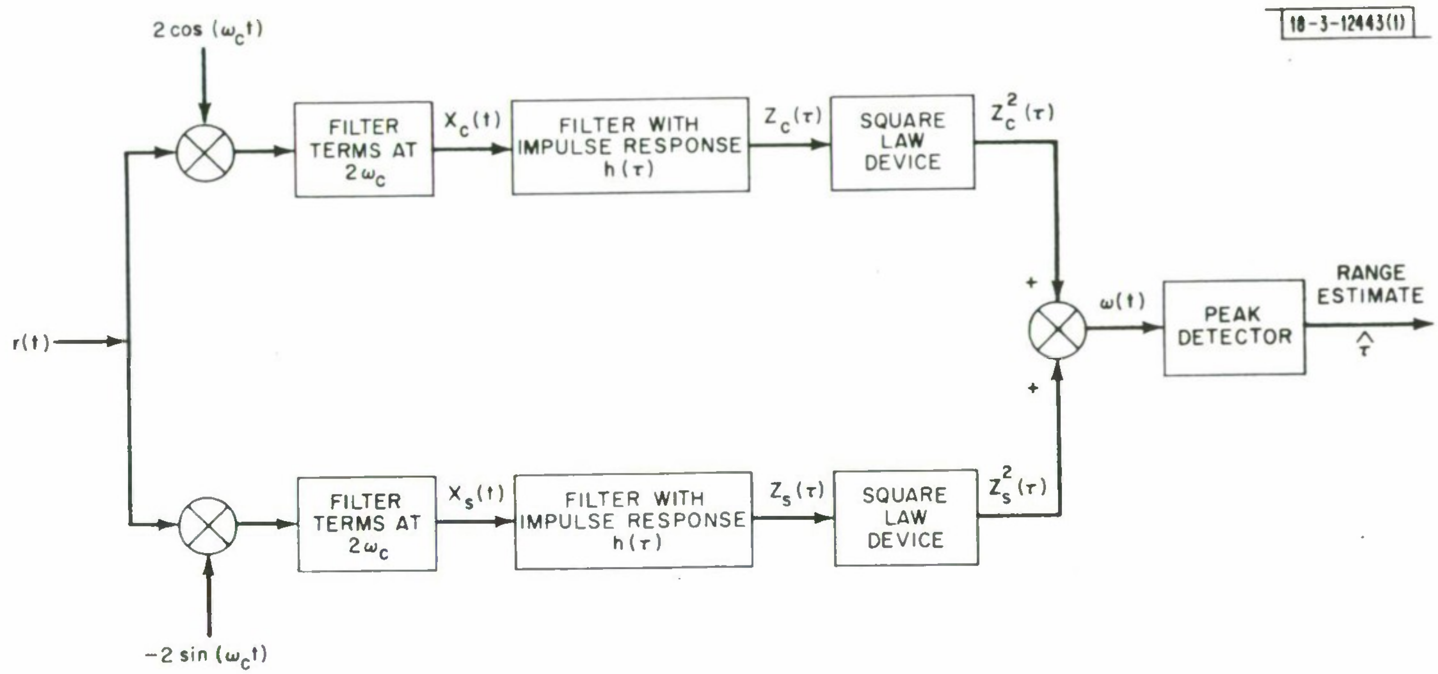

Fig. 1. Signal-processor for range estimation.

$n(t)$ = the complex noise envelope.

We let $h(t)$ be the complex envelope of the impulse response of the filter. Then the signal at the output of the filter is

$$x(\tau) = \int_{-\infty}^{\infty} h(\tau - t)\, r(t)dt$$

$$= Ae^{j\theta} \int_{-\infty}^{\infty} h(\tau - t)p(t-\tau_o)dt + \int_{-\infty}^{\infty} h(\tau - t)n(t)dt \tag{2}$$

We define the quantities

$$\varphi(\tau) = \int_{-\infty}^{\infty} h(\tau - t)p(t)dt \tag{3a}$$

$$\eta(\tau) = \int_{-\infty}^{\infty} h(\tau - t)n(t)dt \tag{3b}$$

and then write for the filter output

$$x(\tau) = Ae^{j\theta}\varphi(\tau - \tau_o) + \eta(\tau) \tag{4}$$

It is at this point that our analysis first begins to digress from that in reference [3]. It is extremely convenient for the work which follows to restrict the class of waveforms and filters to be those for which the filtered signal (which we may sometimes refer to as the compressed pulse) is real. This may appear to be a very restrictive assumption, but in fact it is not. It turns out that after we have solved for the optimum mis-matched filter this assumption is equivalent to requiring that the autocorrelation functions of the input waveforms be real. This in turn will be valid for signals having symmetric spectral densities, which is the case of most practical interest. For example, the linear FM waveform

falls within this class. We could have refrained from making this assumption until a later point in the paper, but we felt that since we were going to make the assumption anyway, we might as well do it earlier in the paper since it makes the manipulations at this point quite a bit easier. Therefore the output of the envelope detector can be written as

$$z(\tau) = |Ae^{j\theta}\varphi(\tau-\tau_o) + \eta(\tau)|^2$$

$$= A^2\varphi^2(\tau-\tau_o) + 2A\varphi(\tau-\tau_o)\mathrm{Re}[e^{j\theta}\eta(\tau)] + |\eta(\tau)|^2 \tag{4}$$

The time-delay corresponding to the target range is taken as the time at which $z(\tau)$ achieves its maximum value. In other words the estimate of τ_o is the number $\hat{\tau}$ where

$$z(\hat{\tau}) = \max_{\tau} z(\tau) \tag{5}$$

In the matched filter case, $h(t) = p^*(-t)$, and in the absence of noise, it follows from (3a) that

$$z(\tau) = [\int_{-\infty}^{\infty} p(t)p^*(t-\tau+\tau_o)dt]^2 \tag{6}$$

which is the square of the autocorrelation function of the transmitted pulse. This quantity is indeed maximized at the true parameter value τ_o. In the mis-matched case and no noise

$$z(\tau) = A^2[\varphi(\tau-\tau_o)]^2 \tag{7}$$

The function $\varphi(\tau)$ will achieve its maximum value at a point τ_b which, in general, may be non-zero. In this case it becomes clear that we should choose as our estimate of τ_o the number $\hat{\tau}_b-\tau_b$, where

$$z(\hat{\tau}_b) = \max_{\tau} z(\tau) \tag{8a}$$

$$\varphi(\tau_b) = \max_{\tau} \varphi(\tau) \tag{8b}$$

It is possible to develope criteria describing the detection, estimation and resolution capabilities of the mis-matched filter in terms of an arbitrary bias τ_b. However it can be shown that τ_b enters the performance equations only through the term $h(t-\tau_b)$. Consequently the performance of the optimum biased filter, $h_b(t)$, can always be achieved by using an unbiased filter $h_u(t)$ by setting $h_u(t) = h_b(t-\tau_b)$, that is, by inserting τ_b units of delay in cascade with $h_b(t)$. Therefore, with no loss in generality we can limit our attention to those filters for which the bias is zero. Hence we need consider only filters for which $\varphi'(0) = 0$, (the prime denotes the derivative with respect to the argument).

The remainder of the analysis proceeds once again along the lines of that described in [3]. We want to point out, however, that the motivation in that paper was to determine the loss in accuracy as a result of using an imperfectly constructed matched filter. No mention is made there of the idea of synthesizing a mis-matched filter to give good range estimates and low sidelobes simultaneously.

We shall now return to the immediate problem of determining the equations describing the range variance of the ad hoc estimator. First we write the output of the envelope detector, $z(\tau)$, as a Taylor series in τ about the point τ_o,

$$z(\tau) = z(\tau_o) + z'(\tau_o)(\tau-\tau_o) + \frac{1}{2} z''(\tau_o)(\tau-\tau_o)^2 + \ldots \tag{9}$$

For large signal-to-noise ratio the estimate $\hat{\tau}$ will be "close to" the true value τ_o and we can expect that

$$z(\hat{\tau}) \doteq z(\tau_o) + z'(\tau_o)(\hat{\tau}-\tau_o) + \frac{1}{2} z''(\tau_o)(\hat{\tau}-\tau_o)^2 \tag{10}$$

But $\hat{\tau}$ is the value which maximizes $z(\tau)$, that is, $z'(\hat{\tau}) = 0$. Therefore it follows that the estimation error is

$$\hat{\tau}-\tau_o = -z'(\tau_o)/z''(\tau_o) \tag{11}$$

We eventually want an expression for the error which is to first order in the noise. This can be obtained by noting that for large SNR the second order noise term in (4) can be neglected so that

$$z(\tau) = A^2\varphi^2(\tau-\tau_o) + 2A\varphi(\tau-\tau_o)\mathrm{Re}[e^{j\theta}\eta(\tau)] \tag{12}$$

and therefore we can write for $z'(\tau)$ the equation

$$z'(\tau) = 2A\varphi'(\tau-\tau_o)\left\{A\varphi(\tau-\tau_o) + \mathrm{Re}[e^{j\theta}\eta(\tau)]\right\} \tag{13}$$

$$+ 2A\varphi(\tau-\tau_o)\mathrm{Re}[e^{j\theta}\eta'(\tau)]$$

Since all filters under consideration are unbiased, $\varphi'(0) = 0$, and therefore

$$z'(\tau_o) = 2A\varphi(0)\mathrm{Re}[e^{j\theta}\eta'(\tau)] \tag{14}$$

This renders the numerator in (11) to be first order in the noise whence it suffices to express the denominator to zero order in the noise. Therefore we have from (13)

$$z''(\tau_o) = 2A^2\varphi(0)\varphi''(0) \tag{15}$$

and the estimation error is

$$\hat{\tau}-\tau_o = -\mathrm{Re}[e^{j\theta}\eta'(\tau_o)]/A\varphi''(0) \tag{16}$$

From (3b) it follows that

$$\eta'(\tau_o) = \int_{-\infty}^{\infty} h'(\tau_o - t)n(t)dt$$

which is a zero mean, Gaussian random variable. For the complex noise envelope one has[1]

$$\overline{n(t)n(s)} = 0$$

$$\overline{n(t)n^*(s)} = 2N_o\,\delta(t-s) \quad {}^{2}$$

whence it follows that

$$\overline{\eta'(\tau_o)\eta'(\tau_o)} = 0 \tag{17a}$$

$$\overline{\eta'(\tau_o)\eta'^*(\tau_o)} = 2N_o \int_{-\infty}^{\infty} |h'(\tau_o - t)|^2 dt \tag{17b}$$

Then if we define the complex variable

$$C = C_x + jC_y = e^{j\theta}\eta'(\tau_o)$$

it follows from (17a) that

$$\overline{C^2} = \overline{C_x^2} - \overline{C_y^2} + 2j\,\overline{C_x C_y} = 0$$

so that $\overline{C_x C_y} = 0$ and $\overline{C_x^2} = \overline{C_y^2}$. Using these facts and (17b) it follows that

$$\overline{CC^*} = \overline{C_x^2} + \overline{C_y^2} = 2\,\overline{C_x^2} = 2N_o \int_{-\infty}^{\infty} |h'(-t)|^2 dt$$

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[1] the bar over a quantity denotes the ensemble average.

[2] N_o is the (single-sided) noise spectral density.

Therefore the estimation error, Equation (16), has zero mean and variance

$$\overline{(\hat{\tau}-\tau_o)^2} = \frac{N_o}{A^2} \cdot \int_{-\infty}^{\infty} |h'(-t)|^2 dt/[\varphi''(0)] \tag{18}$$

From (3a) we find that

$$\varphi''(0) = \int_{-\infty}^{\infty} h''(-t)p(t)dt \tag{19a}$$

$$= - \int_{-\infty}^{\infty} h'(-t)p'(t)dt \tag{19b}$$

where (19b) follows from an integration by parts. Combining (18) and (19) we conclude that for large SNR the mis-matched filter leads to unbiased estimates of the true time delay τ_o. The variance of the estimate is given by

$$\sigma^2_{mmf} = \frac{N_o}{A^2} \cdot \frac{\int_{-\infty}^{\infty} |h'(-t)|^2 dt}{\left[\int_{-\infty}^{\infty} h'(-t)p'(t)dt\right]^2} \tag{20}$$

and this result is valid provided the signal and filter result in a compressed pulse, $\varphi(\tau)$, which is real and has a maximum value at the origin, namely $\varphi'(0) = 0$ or

$$\int_{-\infty}^{\infty} h'(-t)p(t)dt = 0 \tag{21}$$

If the filter is matched to the signal, $h(t) = p^*(-t)$, then (21) is satisfied and (20) becomes

$$\sigma^2_{mf} = N_o / \int_{-\infty}^{\infty} |p'(t)|^2 dt \tag{22}$$

which is the classical result. It is well-known [4] that the matched filter produces the smallest mean-squared range error so that $\sigma^2_{mmf} \geq \sigma^2_{mf}$. Then

$$\frac{\sigma^2_{mmf}}{\sigma^2_{mf}} = \frac{\int_{-\infty}^{\infty} |h'(-t)|^2 dt \int_{-\infty}^{\infty} |p'(t)|^2 dt}{\left[\int_{-\infty}^{\infty} h'(-t)p'(t)\, dt\right]^2} \geq 1 \tag{23}$$

measures the increase in the estimation error as a result of using a mismatched filter.

So far we have considered only the detrimental effects of mismatching in relation to the degradation in range performance as expressed in Equation (23). The major reason for using deliberate mismatching is to achieve better performance in the multiple-target environment. In a typical application the waveform $\varphi(\tau)$ will have, in addition to a maximum at the origin, subsidiary peaks, called sidelobes, at other values of τ. If these sidelobes are not significantly smaller than the mainlobe, then in the multiple-target environment it is possible that targets in adjacent range cells with smaller radar cross-sections will be masked by the sidelobes of a large radar cross-section target. It is desirable, therefore, to design the filter to make these sidelobes small with respect to the magnitude of the central lobe at τ_o. This can be accomplished by requiring that the constraint*

*We can assume that $\varphi(0) > 0$ since $h(t)$ can always be replaced by $-h(t)$ without changing the estimation performance or the zero-bias constraint.

$$-\epsilon(\tau)\varphi(0) \leq \varphi(\tau) \leq \epsilon(\tau)\varphi(0) \qquad (24)$$

be satisfied, where $0 \leq \epsilon(\tau) \leq 1$ represents the sidelobe constraint function and is chosen to combat the particular clutter distribution under consideration. In many practical problem nothing is known concerning the clutter distribution to be encountered and a useful constraint function is to set $\epsilon(\tau) = 1$ for $|\tau| < \delta$, $\epsilon(\tau) = (1-\gamma)$ for $|\tau| > \delta$ where δ is the width of a range resolution cell and γ may be from .9 to .99 in extreme cases. Notice that (21) in conjunction with (24) guarantees that the extremum of $\varphi(\tau)$ at the origin is a global maximum. Henceforth we shall require (with no loss in generality) that

$$\varphi(0) > 0 \qquad (25a)$$

$$-\ddot{\varphi}(0) > 0 \qquad (25b)$$

Using (3) in (24), the constraint on the structure of the compressed pulse will be satisfied provided

$$\left| \int_{-\infty}^{\infty} h(-t)p(t+\tau)dt \right| \leq \epsilon(\tau) \int_{-\infty}^{\infty} h(-t)p(t)dt \qquad (26)$$

In practice h(t) and p(t) will be bandlimited functions, in some sense and therefore "slowly varying" functions of τ. Therefore, the continuum of constraints can be replaced by the finite number of constraints

$$\left| \int_{-\infty}^{\infty} h(-t)p(t+\tau_j)dt \right| \leq \epsilon_j \int_{-\infty}^{\infty} h(-t)p(t)dt \qquad j=1, 2, \ldots, n \qquad (27)$$

where n and $\{\tau_j\}_{j=1}^{n}$ are chosen to provide a dense sampling of the compressed pulse. The problem for which this approximation is not made has been studied recently in reference [5]. It is shown there that the mathematical framework needed to handle the continuum of constraints is so complicated that one not only loses

touch with the physical problem at hand but also finds it impossible to interpret the solution once it is obtained. We shall see that the discretized problem can readily be solved and interpreted and we can then show that as the sampling becomes dense, the sequence of optimum solutions converges with respect to an appropriately defined Hilbert Space norm.

In order to complete the specification of the filter design problem we must take into account the loss of signal detectability which must occur as a result of the filter-signal mismatch. That such a loss occurs follows from the well-known fact that the matched filter results in the largest detection signal-to-noise ratio (SNR). In order to develop a criterion which measures this loss we shall adopt a detection rule which is used in the matched filter processor. That is, a radar target is said to be encountered if

$$z(\hat{\tau}) > \lambda \tag{28}$$

and that no target is present otherwise. In this case $\hat{\tau}$ is the estimate obtained by solving (5) using the processor described above, and λ is a suitably chosen threshold level. Since $\varphi^2(\tau-\tau_o)$, Equation (3), represents the instantaneous signal power at the output of the filter, and since

$$2N_o \int_{-\infty}^{\infty} |h(-t)|^2 dt \tag{29}$$

represents the average noise power, the detection noise-to-signal ratio is measured by the quantity

$$2N_o \int_{-\infty}^{\infty} |h(-t)|^2 dt / \varphi^2(\hat{\tau}-\tau_o) \tag{30}$$

A well-designed receiver will provide good estimates of τ_o so that $\hat{\tau}-\tau_o \approx 0$ and the detection performance of the mismatched filter is described by the noise-to-signal ratio

$$\rho_{mmf} = 2N_o \frac{\int_{-\infty}^{\infty} |h(-t)|^2 \, dt}{\left[\int_{-\infty}^{\infty} h(-t)p(t)dt\right]^2} \tag{31}$$

for the matched filter case, $h(-t) = p^*(t)$, and we have

$$\rho_{mf} = 2N_o / \int_{-\infty}^{\infty} |p(t)|^2 dt \tag{32}$$

The ratio

$$\frac{\rho_{mmf}}{\rho_{mf}} = \frac{\int_{-\infty}^{\infty} |h(-t)|^2 \, dt \int_{-\infty}^{\infty} |p(t)|^2 dt}{\left[\int_{-\infty}^{\infty} h(-t)p(t)dt\right]^2} \geq 1 \tag{33}$$

represents the increase in the detection noise-to-signal ratio as a result of using a mismatched filter. Since the signal $p(t)$ is specified we can define

$$\int_{-\infty}^{\infty} |p(t)|^2 \, dt = E \tag{34a}$$

$$\int_{-\infty}^{\infty} |p'(t)|^2 dt = B^2 \tag{34b}$$

E represents the signal energy which B^2/E represents its mean square bandwidth.

On the basis of the preceding analysis we can combine Equations (21), (23), (27), (33), and (34) to formulate two important filter design problems: From the class of admissible filters, we want to find that filter, which, for a given signal $p(t)$, satisfies the multiple target resolution constraints, Equation (27).

$$\left| \int_{-\infty}^{\infty} h(-t)p(t+\tau_j)dt \right| \leq \epsilon_j \int_{-\infty}^{\infty} h(-t)p(t)dt \qquad j=1, 2, \ldots, n \tag{35}$$

and satisfies the zero bias constraint, Equation (21)

$$\int_{-\infty}^{\infty} h'(t)p(t)dt = 0 \tag{36}$$

and extremizes the cost functionals for the following problems:

Problem 1: minimizes the normalized detection noise-to-signal ratio, Equation (33)

$$E \cdot \frac{\int_{-\infty}^{\infty} |h(-t)|^2 dt}{\left[\int_{-\infty}^{\infty} h(-t)p(t)dt\right]^2} \tag{37}$$

Problem 2: minimizes the normalized increase in the mean-square range estimation error Equation (23)

$$B^2 \cdot \frac{\int_{-\infty}^{\infty} |h'(-t)|^2 dt}{\left[\int_{-\infty}^{\infty} h'(-t)p'(t)dt\right]^2} \tag{38}$$

subject to a constraint on the allowable increase in detection noise-to-signal Equation (33)

$$E \cdot \frac{\int_{-\infty}^{\infty} |h(-t)|^2 dt}{\left[\int_{-\infty}^{\infty} h(-t)p(t)dt\right]^2} = \beta > 1 \tag{39}$$

It is advantageous to combine these two problems into a one-parameter family of optimization problems which minimizes a linear combination of the detection and estimation performance indices. Therefore, we combine Equations (37), and (38) and (39) and minimize

$$\alpha \frac{B^2}{E} \frac{\int_{-\infty}^{\infty} |h'(-t)|^2 dt}{\left[\int_{-\infty}^{\infty} h'(-t)p'(t)dt\right]^2} + (1-\alpha) \frac{\int_{-\infty}^{\infty} |h(-t)|^2 dt}{\left[\int_{-\infty}^{\infty} h(-t)p(t)dt\right]^2} \text{ for } 0 \leqq \alpha \leqq 1 \tag{40}$$

subject to the zero bias constraint, Equation (36)

$$\int_{-\infty}^{\infty} h'(-t)p(t)dt = 0 \tag{41}$$

and sidelobe constraints, Equation (35)

$$-\epsilon_j \int_{-\infty}^{\infty} h(-t)p(t+\tau_j)dt \leqq \int_{-\infty}^{\infty} h(-t)p(t)dt \leqq \epsilon_j \int_{-\infty}^{\infty} h(-t)p(t+\tau_j)dt$$

$$j=1, 2, \ldots, n \tag{42}$$

By setting $\alpha = 0$ we could obtain the filter which maximizes the detection signal-to-noise ratio subject to sidelobe constraints. This might be a useful solution to obtain for some communications problems in which the signal must "stand out" of the noise. For $\alpha = 1$ the solution would yield a filter which gives the best range accuracy subject to sidelobe constraints. It would be tempting to use this filter for pulse position modulation communications problems and for the radar tracking problem. However, one intuitively would expect the performance to depend on the ability to detect the target, since if the signal does not "stand out" of the noise, it would not be possible to measure the location of the peak of the compressed pulse. In fact we shall show that the case $\alpha = 1$ leads to a mathematically ill-defined problem. In addition, the formula describing the estimation performance of the filter implicitly assumed a large signal-to-noise ratio. This assumption can not be guaranteed unless $\alpha < 1$.

For the cases in which $0 < \alpha < 1$, we are merely putting different weights on the detection and estimation performance characteristics of the filter. By solving the problem for several values of α we can choose the filter which

gives an acceptable detection performance and good estimation performance. It can be shown that that these solutions correspond to those of Problem 2 for appropriate values of β. Therefore, the problem which is described by Equations (40), (41), and (42) describes a wide class of problems which arise in radar and communications. It is to this latter problem formulation that we shall direct our attention in the following sections. In the next section we shall reformulate this problem in state-space and by using optimal control techniques we shall obtain necessary conditions for the optimum filter. From these conditions we can deduce the physical configuration of the optimum filter. However, certain mathematical difficulties arise in the time domain formulation which are difficult to resolve. By transforming the problem to the frequency domain and using the insights obtained from the structure of the time domain solution, it is possible to use functional analysis to obtain the Fourier Transform of the unique optimum filter. This solution specifies the optimum filter except for certain unknown gains. The cost functional, the zero bias constraint and the sidelobe constraints can be evaluated in terms of these unknown constants which in turn leads to a nonlinear programming problem. Efficient computational algorithms which have been developed for this type of problem are then used to generate numerical results to illustrate the utility of this approach to filter design.

III. State-Space Solution of the Design Problem

In this section we shall formulate and solve the mismatched filter design problem using state-variable techniques. Most of the section involves straightforward details which will provide no new insights to anyone familiar with optimal control theory. It is included at this point in the paper because this was the order in which we performed our research. It shows that by following the recipe prescribed by the minimum principle of Pontryagin one can formally obtain the structure of the optimum filter. With the solution before us, we were then able to obtain a much more rigorous derivation of the result which uses only the projection theorem of Hilbert Space Theory. The latter solution is presented in Section IV. The point is that Section III can be left out by the casual reader without significant loss in continuity. It is included for those who may have an interest in the methodology which must be followed to use optimal control theory to solve optimization problems. Some limitations of the method are pointed out.

In order to guarantee that the optimal control approach will result in a realizable impulse response, we shall replace the function h(-t) which appears in all of our equations by h(T-t) where T denotes some suitably large processing time. We shall assume that h(t) and p(t) are real. In addition, we assume that p(t) and $p(t+\tau_j)$, j=;, . . . , n are zero outside of the interval [0, T].

We now proceed to the state-space formulation by defining the control function to be the first derivative of the impulse response and let the first

component of the state vector be the impulse response*, i.e.

$$u(t) = \dot{h}(T-t) \tag{46a}$$

$$x_1(t) = h(T-t) \tag{46b}$$

Then the first state equation is

$$\dot{x}_1(t) = -u(t) \tag{47}$$

In order that $\dot{h}(T-t)$ be well defined for all $t\epsilon[0, T]$ we require that $x_1(0) = x_1(T) = 0$. In addition we define the state variables

$$\dot{x}_2(t) = \dot{p}(t)\dot{h}(T-t) \tag{48a}$$

$$\dot{x}_3(t) = p(t)\dot{h}(T-t) \tag{48b}$$

$$\dot{x}_4(t) = h^2(T-t) \tag{48c}$$

$$\dot{x}_5(t) = \dot{h}^2(T-t) \tag{48d}$$

$$\dot{y}_o(t) = p(t)h(T-t) \tag{48e}$$

$$\dot{y}_j(t) = p(t+\tau_j)h(T-t) \quad j=1, 2, \ldots, n \tag{48f}$$

Each of these variables is to have zero initial conditions, whence it follows that

$$x_2(T) = \int_o^T \dot{p}(t)\dot{h}(T-t)dt \tag{49a}$$

$$x_3(T) = \int_o^T p(t)\dot{h}(T-t)dt \tag{49b}$$

$$x_4(T) = \int_o^T [h(T-t)]^2 dt \tag{49c}$$

$$x_5(T) = \int_o^T [\dot{h}(T-t)]^2 dt \tag{49d}$$

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* The dot denotes differentiation with respect to time.

$$y_o(T) = \int_o^T p(t)h(T-t)dt \tag{49e}$$

$$y_j(T) = \int_o^T p(t+\tau_j)h(T-t)dt \tag{49f}$$

$$j=1, 2, \ldots, n$$

Notice that these are all of the quantities needed to specify the filter design problem. By substituting Equation (46) into Equation (48) we obtain the state equations

$$\dot{x}_1(t) = -u(t) \qquad x_1(0) = 0 \qquad x_1(T) = 0 \tag{50a}$$

$$\dot{x}_2(t) = \dot{p}(t)u(t) \qquad x_2(0) = 0 \tag{50b}$$

$$\dot{x}_3(t) = p(t)u(t) \qquad x_3(0) = 0 \tag{50c}$$

$$\dot{x}_4(t) = x_1^2(t) \qquad x_4(0) = 0 \tag{50d}$$

$$\dot{x}_5(t) = u^2(t) \qquad x_5(0) = 0 \tag{50e}$$

$$\dot{y}_o(t) = p(t)x_1(t) \qquad y_o(0) = 0 \tag{50f}$$

$$\dot{y}_j(t) = p(t+\tau_j)x_1(t) \qquad y_j(0) = 0 \quad j=1, 2, \ldots, n \tag{50g}$$

The resolution constraints, Equation (42), require that

$$-\epsilon_j y_o(T) \leqq y_j(T) \leqq \epsilon_j y_o(T) \quad j=1, 2, \ldots, n \tag{51}$$

and the zero bias constraint, Equation (41), requires that $x_3(T) = 0$. The cost functional to be minimized, from Equation (40), can be written as

$$P[\underline{x}(T)] = \alpha \cdot \frac{x_5(T)}{x_2^2(T)} \cdot \frac{B^2}{E} + (1-\alpha) \cdot \frac{x_4(T)}{y_o^2(T)} \tag{52}$$

Since all of the equations are homogeneous in $h(\cdot)$, we can normalize either $x_2(T)$ or $y_o(T)$. Therefore, the filter design problem is equivalent to finding the control function $u(t)$, $t\epsilon[0, T]$ which minimizes the cost functional

$$P[\underline{x}(T)] = \alpha \cdot \frac{x_5(T)}{x_2^2(T)} \cdot \frac{B^2}{E} + (1-\alpha)\, x_4(T) \tag{53}$$

subject to the differential equation constraints

$$\dot{x}_1(t) = -u(t) \qquad x_1(0) = 0 \qquad x_1(T) = 0 \tag{54a}$$

$$\dot{x}_2(t) = \dot{p}(t)u(t) \qquad x_2(0) = 0 \tag{54b}$$

$$\dot{x}_3(t) = p(t)u(t) \qquad x_3(0) = 0 \qquad x_3(T) = 0 \tag{54c}$$

$$\dot{x}_4(t) = x_1^2(t) \qquad x_4(0) = 0 \tag{54d}$$

$$\dot{x}_5(t) = u^2(t) \qquad x_5(0) = 0 \tag{54e}$$

$$\dot{y}_o(t) = p(t)x_1(t) \qquad y_o(0) = 0 \qquad y_o(T) = 1 \tag{54f}$$

$$\dot{y}_j(t) = p(t+\tau_j)x_1(t) \qquad y_j(0) = 0 \qquad j=1, 2, \ldots, n \tag{54g}$$

subject to the resolution constraints

$$-\epsilon_j \leqq y_j(T) \leqq \epsilon_j \qquad j=1, 2, \ldots, n \tag{55}$$

Although the dynamics of the system equations are relatively simple, the inequality end-point constraints, Equation (55), makes this an interesting optimal control problem. We shall deal with these constraints using the method suggested in [6] in which we imbed the original problem into a subclass of problems in which the terminal conditions are fixed. Therefore, we shall solve a fixed end-point problem by requiring that

$$y_j(T) = y_j^f \qquad j=1, 2, \ldots, n \tag{56}$$

where the n-vector $\underline{y}^f$ belongs to a set $S \subset E^n$ where

$$S = \{\underline{y}^f: \ -\epsilon_j \leq y_j^f \leq \epsilon_j, \quad j=1,2,\ldots,n\} \tag{57}$$

For each vector $\underline{y}^f \in S$ we have a fixed terminal time optimal control problem for which the necessary conditions for optimality can be derived using the minimum principle [7]. Consider the general control system described by

$$\dot{\underline{z}}(t) = \underline{f}[\underline{z}(t), u(t), t] \qquad 0 \leq t \leq T \tag{58a}$$

$$\underline{z}(0) = \underline{z}^o \tag{58b}$$

$$g_j[\underline{z}(T)] = 0 \qquad j=1,2,\ldots,r \tag{58c}$$

in which a minimum of $P[\underline{z}(T)]$ is sought, we first form the Hamiltonian function

$$H[\underline{z}(t), \underline{\lambda}(t), u(t), t] = \sum_{j=1}^{N} \lambda_j(t) f_j[\underline{z}(t), u(t), t] \tag{59}$$

where the costate variable $\underline{\lambda}(t)$ satisfies the differential equation

$$\dot{\lambda}_j(t) = -\frac{\partial H}{\partial z_j(t)} \qquad j=1,2,\ldots,n \tag{60a}$$

with end-conditions

$$\lambda_j(T) = \frac{\partial P[\underline{z}(T)]}{\partial z_j(T)} + \sum_{i=1}^{r} \alpha_i \frac{\partial g_i[\underline{z}(T)]}{\partial z_j(T)} \tag{60b}$$

The constants α_j are unknown multipliers which are to be chosen so that the prescribed end-conditions, Equation (58c), will be satisfied.

Applied to the problem at hand the Hamiltonian is,

$$\begin{aligned} H = {} & \lambda_1(t)u(t) + \lambda_2(t)\dot{p}(t)u(t) + \lambda_3(t)p(t)u(t) \\ & + \lambda_4(t)x_1^2(t) + \lambda_5(t)u^2(t) + \mu_o(t)p(t)x_1(t) \\ & + \sum_{k=1}^{n} \mu_k(t)p(t+\tau_k)x_1(t) \end{aligned} \tag{61}$$

Therefore, the costate variables satisfy the equations

$$\dot{\lambda}_1(t) = -2\lambda_4(t)x_1(t) - \mu_o(t)p(t) - \sum_{k=1}^{n} \mu_k(t)p(t+\tau_k) \tag{62a}$$

$$\dot{\lambda}_j(t) = 0 \qquad j=2,\ldots,5 \tag{62b}$$

$$\dot{\mu}_j(t) = 0 \qquad j=0,1,\ldots,n \tag{62c}$$

The end-point constraint functions are

$$g_1[\underline{z}(T)] = x_1(T) \tag{63a}$$

$$g_2[\underline{z}(T)] = x_3(T) \tag{63b}$$

$$g_3[\underline{z}(T)] = y_o(T) - 1 \tag{63c}$$

$$g_{3+j}[\underline{z}(T)] = y_j(T) - y_j^f \qquad j=1,2,\ldots,n \tag{63d}$$

and the cost functional is

$$P[\underline{z}(T)] = \alpha \cdot \frac{x_5(T)}{x_2^2(T)} \cdot \frac{B^2}{E} + (1-\alpha)\, x_4(T) \tag{64}$$

Applied to Equation (60b), the terminal values of the costate variables are

$$\lambda_1(T) = \alpha_1 \tag{65a}$$

$$\lambda_2(T) = -2\alpha \frac{x_5(T)}{x_2^3(T)} \cdot \frac{B^2}{E} \tag{65b}$$

$$\lambda_3(T) = \alpha_2 \tag{65c}$$

$$\lambda_4(T) = (1-\alpha) \tag{65d}$$

$$\lambda_5(T) = \alpha \cdot \frac{1}{x_2^2(T)} \cdot \frac{B^2}{E} \tag{65e}$$

$$\mu_o(T) = \alpha_3 \tag{65f}$$

$$\mu_j(T) = \alpha_{3+j} \qquad j=1,2,\ldots,n \tag{65g}$$

From Equation (62b), (62c), it is clear that all costate variables except $\lambda_1(\cdot)$ are constants. Therefore, the Hamiltonian can be written as

$$H = [-\lambda_1(t) - 2\alpha \cdot \frac{x_5(T)}{x_2^3(T)} \cdot \frac{B^2}{E} \dot{p}(t) + \alpha_2 p(t)]\, u(t)$$

$$+ \frac{\alpha}{x_2^2(T)} \cdot \frac{B^2}{E} \cdot u^2(t) + (1-\alpha) x_1^2(t) + \alpha_3 p(t) x_1(t)$$

$$+ \sum_{k=1}^{n} \alpha_{3+k}\, p(t+\tau_k)\, x_1(t) \tag{66}$$

The minimum principle states that the optimal control must minimize H at each $t \in [0, T]$. The minimization is performed over the class of admissible controls, which in turn, is directly related to the class of admissible filters. If any "buildability" constraints are to be incorporated into the synthesis, it is at this point that they are to be taken into account. For the purpose of this investigation, however, we shall assume that there are no constraints on the filter's structure. Then Equation (66) can be minimized over the variable $u(t)$ for each $t \in [0, T]$ by minimizing

$$\tilde{H}[u(t)] = [-\lambda_1(t) - 2\alpha \frac{x_5(T)}{x_2^3(T)} \cdot \frac{B^2}{E} \dot{p}(t) + \alpha_2 p(t)]\, u(t)$$

$$+ \frac{\alpha}{x_2^2(T)} \cdot \frac{B^2}{E} u^2(t) \tag{67}$$

Recall the fact that the weighting constant $\alpha \epsilon [0, 1]$. For $0 < \alpha \leq 1$, the function H has a well-defined minimum which is achieved by the control

$$u^*(t) = \frac{x_2^2(T)}{2\alpha} \cdot \frac{E}{B^2} \cdot \left[\lambda_1(t) + 2\alpha \frac{x_5(T)}{x_2^3(T)} \cdot \dot{p}(t) - \alpha_2 p(t)\right] \qquad (68)$$

However if $\alpha = 0$, then the control is singular [8] provided

$$\lambda_1(t) - \alpha_2 p(t) \equiv 0 \qquad \text{for all } t \epsilon [0, T] \qquad (69)$$

Otherwise the optimum control switches from $\pm\infty$ which is a meaningless solution. Information regarding the nature of the optimum singluar control can be obtained by differentiating Equation (69) and relating the result to Equation (62a). We obtain

$$\begin{aligned} \alpha_2 \dot{p}(t) &\equiv \dot{\lambda}_1(t) \\ &= -2x_1^*(t) - \mu_o p(t) - \sum_{k=1}^{n} \mu_k p(t+\tau_k) \end{aligned} \qquad (70)$$

where we have set $\mu_k = \alpha_{3+k}$, $k=0, 1, \ldots, n$. But by definition $x_1(t) = h(T-t)$, therefore, the optimum impulse response is of the form

$$h^*(T-t) = \lambda_1 \dot{p}(t) + \lambda_2 p(t) + \sum_{k=1}^{n} \mu_k p(t \pm \tau_k) \qquad (71)$$

where $\lambda_1, \lambda_2, \mu_1, \ldots, \mu_n$ are arbitrary constants. These are to be chosen to satisfy the sidelobe and zero-bias constraints and to further minimize the cost function. We shall discuss this idea at length at a later time. The point we want to make here is that the optimal control methodology has led us to the form of the optimum filter, at least for the case $\alpha = 0$. But setting $\alpha = 0$ in the cost functional, Equation (53), means that no consideration was being given to the range estimate variance; only the detection noise-to-signal ratio was being minimized. Therefore, the singular control leads to the structure of the

filter which maximizes the detection signal-to-noise ratio subject to fixed constraints on the sidelobe structure. Equation (71), therefore, proves that the transversal equalizer is the optimum filter for maximizing the detection SNR in a multiple target environment.

Next we return to the case $0<\alpha\leq 1$. For this case the optimal control is given by Equation (68). However, since $\dot{x}_1(t) = -u(t)$, then Equation (68) can be written as

$$\dot{x}_1^*(t) = -\frac{E}{B^2}\,\frac{x_2^2(T)}{2\alpha}\cdot\lambda_1(t) - \frac{x_5(T)}{x_2(T)}\,\dot{p}(t) + \frac{E}{B^2}\,\frac{x_2^2(T)}{2\alpha}\,p(t) \tag{72}$$

We define a new function according to the equation

$$x_1^*(t) = e(t) - \frac{x_5(T)}{x_2(T)}\,p(t) \tag{73}$$

which, when substituted in Equation (72) leads to

$$\dot{e}(t) = -\frac{E}{B^2}\cdot\frac{x_2^2(T)}{2\alpha}\cdot\lambda_1(t) + \frac{E}{B^2}\,\frac{x_2^2(T)}{2\alpha}\,\alpha_2 p(t) \tag{74}$$

Since the derivatives of $\lambda_1(\cdot)$ and $p(\cdot)$ have been assumed to exist, then

$$\ddot{e}(t) = -\frac{E}{B^2}\cdot\frac{x_2^2(T)}{2\alpha}\cdot\dot{\lambda}_1(t) + \frac{E}{B^2}\,\frac{x_2^2(T)}{2\alpha}\,\alpha_2\dot{p}(t) \tag{75}$$

and then using Equation (62a) for $\dot{\lambda}_1(\cdot)$, we have

$$\ddot{e}(t) = -\frac{E}{B^2}\,\frac{x_2^2(T)}{2\alpha}\left[-2(1-\alpha)x_1^*(t) - u_o p(t) - \sum_{k=1}^{n} u_k p(t+\tau_k)\right] + \frac{E}{B^2}\,\frac{x_2^2(T)}{2\alpha}\,\alpha_2\dot{p}(t) \tag{76}$$

Then utilizing Equation (73) for $x_1(t)$ we have the second order differential equation for $e(\cdot)$,

$$\ddot{e}(t) - \frac{(1-\alpha)}{\alpha} \frac{E}{B^2} x_2^{\,2}(T) e(t) = \alpha_2 \dot{p}(t) + \mu_o p(t) + \sum_{k=1}^{n} \mu_k p(t+\tau_k) \tag{77}$$

and this equation is to be solved subject to the end-condition, $e(0) = e(T) = 0$, which follows from Equation (73). Assuming that we can solve this equation for $e(\cdot)$, the optimum filter impulse response for the combined estimation and detection problems is

$$h^*(T-t) = e(t) - \frac{x_5(T)}{x_2(T)} p(t) \tag{78}$$

We let

$$\alpha_1 = - x_5(T)/x_2(T) \tag{79a}$$

$$\omega^2 = \frac{(1-\alpha)}{\alpha} \frac{E}{B^2} x_2^{\,2}(T) \tag{79b}$$

It is clear that ω^2 is a positive constant. It follows from (25) and the definitions of $x_5(T)$ and $x_2(T)$ that α_1 must be positive. Then the optimum filter impulse response is

$$h^*(T-t) = e(t) + \alpha_1 p(t) \qquad \alpha_1 > 0 \tag{80a}$$

where $e(t)$ satisfies the equation

$$\ddot{e}(t) - \omega^2 e(t) = \alpha_2 \dot{p}(t) + \mu_o p(t) + \sum_{k=1}^{n} \mu_k p(t+\tau_k) \tag{80b}$$

$$e(0) = e(T) = 0 \tag{80c}$$

There remains the problem of selecting the multipliers $\alpha_1, \alpha_2, \mu_o, \mu_1, \ldots, \mu_n$ in order to guarantee that the specified constraints are satisfied. The discussion of this problem will be deferred until a later time. The main point to make here is that the optimum filter is made up of parallel processors: in one path is a

matched filter, in another a matched filter and transversal equalizer are cascaded with some sort of unstable dynamical system. For a fixed processing time a well-defined solution results, but for the case in which $T \to \infty$, an unstable mode arises in the filter's realization. Furthermore, we really are interested in obtaining a filter impulse response on $[0, \infty)$ and this is not possible using the state-space technique directly. Presumably we could use the method outlined above to solve the problem for each T_n and let $T_n \to \infty$ but it is extremely difficult to study the behavior of the optimum solutions as a function of T_n except in the simplest case of minimizing the detection noise-to-signal ratio. Therefore, there is a good reason to re-examine the filter design problem in the frequency domain where the problems of stability and fixed terminal time do not arise. We do not mean to imply that the optimal control formulation has been useless, since by following the recipe prescribed by the minimum principle we have succeeded in deducing the structure of the optimum filters. This information indicates the appropriate inner product space to use when solving the problem in the frequency domain.

IV. Frequency Domain Solution of the Design Problem

In this section we shall use the $L^2(-\infty, \infty)$ theory of Fourier transforms to transform the filter design problem described in Equation (40), (41), and (42) into the frequency domain. Therefore we need assume that $h(\cdot)$, $\dot{h}(\cdot)$, $p(\cdot)$, and $\dot{p}(\cdot)$ are members of the space

$$L^2(-\infty, \infty) = \{ f(\cdot): \int_{-\infty}^{\infty} |f(t)|^2 dt < \infty \} \qquad (81)$$

where $f(.)$ may be a complex-valued function. Then the Fourier Transform of $f(\cdot)$, is well-defined by

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \triangleq \mathfrak{F}[f(t)] \qquad (82a)$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \triangleq \mathfrak{F}^{-1}[F(\omega)] \qquad (82b)$$

where the "=" signs mean "limit in the mean". We shall have occasion to use Parseval's Theorem

$$\int_{-\infty}^{\infty} f(t) g(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) G^*(\omega) d\omega \qquad (83)$$

We also note that if $F(\omega) = \mathfrak{F}[f(t)]$, then $j\omega F(\omega) = \mathfrak{F}[\dot{f}(t)]$ and that $F^*(\omega) = \mathfrak{F}[f^*(-t)]$.

Using these relations, the filter design problem stated in Equations (40), (41), and (42) becomes the problem of minimizing

$$\alpha \frac{B^2}{E} : \frac{\int_{-\infty}^{\infty} \omega^2 |H(\omega)|^2 d\omega}{[\int_{-\infty}^{\infty} \omega^2 H(\omega) P(\omega) d\omega]^2} + (1-\alpha) \frac{\int_{-\infty}^{\infty} |H(\omega)|^2 d\omega}{[\int_{-\infty}^{\infty} H(\omega) P(\omega) d\omega} \qquad (84)$$

$$0 \leq \alpha \leq 1$$

subject to the zero-bias constraint

$$\int_{-\infty}^{\infty} j\omega H(\omega) P(\omega) d\omega = 0 \tag{85}$$

and sidelobe constraints

$$-\epsilon_k \int_{-\infty}^{\infty} H(\omega)P(\omega)d\omega \leq \int_{-\infty}^{\infty} H(\omega)P(\omega)e^{j\omega\tau_k} d\omega \leq \epsilon_k \int_{-\infty}^{\infty} H(\omega)P(\omega)d\omega \tag{86}$$

$$k=1, 2, \ldots, n$$

The inequality constraints are as troublesome in the frequency domain as they were in the time domain. However. we can use the same trick to eliminate them and first solve a problem with equality constraints. With no loss of generality we can set

$$\int_{-\infty}^{\infty} \omega^2 H(\omega)P(\omega)\frac{d\omega}{2\pi} = B^2 \tag{87}$$

because the equations are homogeneous in $H(\omega)$. Furthermore we require that

$$\int_{-\infty}^{\infty} H(\omega)P(\omega)e^{j\omega\tau_k}\frac{d\omega}{2\pi} = y_k \qquad k=0, 1, \ldots, n \tag{88}$$

where $\tau_o = 0$ and the (n+1)-vector $\underline{y}$ belongs to the $S \subset E^{n+1}$ where

$$S = \{\underline{y}: \ -\epsilon_k y_o \leq y_k \leq \epsilon_k y_o, \quad k=1, 2, \ldots, n\} \tag{89}$$

If we consider only those cases for which $\alpha > 0$, then for each fixed $\underline{y} \in S$ we want to minimize

$$\frac{\alpha}{B^2 E} \cdot \left[\int_{-\infty}^{\infty} \omega^2 |H(\omega)|^2 d\omega + \frac{(1-\alpha)}{\alpha} \cdot \frac{B^2 E}{y_o^2} \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega\right] \tag{90}$$

subject to the constraints

$$\int_{-\infty}^{\infty} j\omega H(\omega)P(\omega)d\omega = 0 \tag{91}$$

$$\int_{-\infty}^{\infty} \omega^2 H(\omega) P(\omega) \frac{d\omega}{2\pi} = B^2 \tag{92}$$

$$\int_{-\infty}^{\infty} H(\omega) P(\omega) \frac{d\omega}{2\pi} = y_o \tag{93}$$

$$\int_{-\infty}^{\infty} H(\omega) P(\omega) e^{j\omega\tau_k} \frac{d\omega}{2\pi} = y_k \quad k=1, 2, \ldots, n \tag{94}$$

From Parsevals Theorem we know that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |f(t)|^2 dt \tag{95}$$

Therefore since $h(\cdot)$, $\dot{h}(\cdot)$, $p(\cdot)$, $\dot{p}(\cdot)$ belongs to $L^2(-\infty, \infty)$ then $H(\omega)$, $\omega H(\omega)$, $P(\omega)$, $\omega P(\omega)$ belong to $L^2(-\infty, \infty)$. Let us now define a vector space $\mathcal{M}$ over the real numbers as

$$\mathcal{M} = \{ F(\omega): \; F(\omega), \; \omega F(\omega) \in L^2(-\infty, \infty), \; F(\omega)P(\omega) = F^*(-\omega)P^*(-\omega)\} \tag{96}$$

The condition $F(\omega)P(\omega) = F^*(-\omega)P^*(-\omega)$ is equivalent to the requirement that the complex representation of the compressed pulse be real. We define an inner product on $\mathcal{M}$ as follows: For any $F(\cdot)$ and $G(\cdot) \in \mathcal{M}$,

$$\langle F(\omega), G(\omega) \rangle = \int_{-\infty}^{\infty} [\omega^2 + \frac{(1-\alpha)}{\alpha} \cdot \frac{B^2 E}{y_o^2}] \, F(\omega) G^*(\omega) \frac{d\omega}{2\pi} \tag{97}$$

For convenience we set

$$\lambda^2 = \frac{(1-\alpha)}{\alpha} \cdot \frac{B^2 E}{y_o^2} \tag{98}$$

and note that this is a positive quantity. It is easy to verify that Equation (97) specifies a well-defined inner product. Furthermore it can be verified that the pair $(\mathcal{M}, \langle \cdot, \cdot \rangle)$ defines a Hilbert Space. We can then imbed the minimization problem into this Hilbert Space and minimize Equation (90) which becomes

$$\frac{\alpha}{B^2 E} \|H(\omega)\|^2 \tag{99}$$

where $\|F(\omega)\|^2 = \langle F(\omega), F(\omega) \rangle$. The minimization performed subject to the constraints, Equations (91) through (94), which can now be written as

$$\langle H(\omega), \frac{j\omega}{\omega^2+\lambda^2} P^*(\omega) \rangle = 0 \tag{100}$$

$$\langle H(\omega), \frac{\omega^2}{\omega^2+\lambda^2} P^*(\omega) \rangle = B^2 \tag{101}$$

$$\langle H(\omega), \frac{1}{\omega^2+\lambda^2} P^*(\omega) \rangle = y_o \tag{102}$$

$$\langle H(\omega), \frac{1}{\omega^2+\lambda^2} P^*(\omega) e^{-j\omega\tau_k} \rangle = y_k \quad k=1, 2, \ldots, n \tag{103}$$

These operations are well-defined provided the functions appearing in the inner product belong to the space $\mathcal{M}$. Since $H(\omega)$ and $P(\omega)$ belong to $\mathcal{M}$ by assumption, it is easy to show that the functions appearing in the above inner products belong to $\mathcal{M}$ as long as $\lambda \neq 0$. Notice that $\lambda = 0$ if and only if $\alpha = 1$. Referring to Equation (84), this case corresponds to minimizing the estimate variance subject to no constraints on the detection signal-to-noise ratio. We have already given physical arguments based on the practical problem for requiring that $\alpha \neq 1$. Now we have a mathematical requirement that this condition be met.

Next we define a set L to be

$$L = \{F(\omega): \ F(\omega) = a_1 \frac{j\omega}{\omega^2+\lambda^2} P^*(\omega) + a_2 \frac{\omega^2}{\omega^2+\lambda^2} P^*(\omega) + \mu_o \frac{1}{\omega^2+\lambda^2} P^*(\omega) + \sum_{k=1}^{n} \mu_k \frac{1}{\omega^2+\lambda^2} P^*(\omega) e^{-j\omega\tau_k}\} \tag{104}$$

where a_1, a_2, μ_o, μ_1, ..., μ_n are arbitrary real constants. It is easy to show that L is a subspace of $\mathcal{M}$. Then the projection theorem for functions in a Hilbert Space holds [14] and we can express the function $H(\omega) \in \mathcal{M}$ as follows

$$H(\omega) = H_L(\omega) + H_\perp(\omega) \tag{105}$$

where

$$H_L(\omega) \in L \text{ and}$$

$$< H_\perp(\omega), \; F(\omega) > = 0 \text{ for all } F(\omega) \in L \tag{106}$$

This decomposition of $H(\omega)$ is unique. Then from Equation (99) the function to be minimized is

$$\frac{\alpha}{B^2 E} \|H_L(\omega)\| + \frac{\alpha}{B^2 E} \|H_\perp(\omega)\|^2 \tag{107}$$

All of the functions appearing in the constraint equations, Equations (100) through (103), belong to the subspace L and are therefore orthogonal to $H_\perp(\omega)$. (Therefore only that part of $H(\omega)$ belonging to L is utilized in satisfying the constraints.) Hence, the problem reduces to minimizing (107) subject to the constraints.

$$< H_L(\omega), \frac{j\omega}{\omega^2+\lambda^2} P^*(\omega) > = 0 \tag{108}$$

$$< H_L(\omega), \frac{\omega^2}{\omega^2+\lambda^2} P^*(\omega) > = B^2 \tag{109}$$

$$< H_L(\omega), \frac{1}{\omega^2+\lambda^2} P^*(\omega) > = y_o \tag{110}$$

$$< H_L(\omega), \frac{1}{\omega^2+\lambda^2} P^*(\omega) e^{-j\omega\tau_k} > = y_k \quad k=1, 2 \; ..., n \tag{111}$$

It is obvious that the norm can be made smallest by choosing $H_{\perp}(\omega) \equiv 0$. Then the optimum solution $\hat{H}(\omega)$ belongs to L and therefore is of the form

$$\hat{H}(\omega) = \frac{1}{\omega^2+\lambda^2}[a_1 j\omega + a_2\omega^2 + u_o + \sum_{k=1}^{n} u_k e^{-j\omega\tau_k}]P^*(\omega) \tag{112}$$

Since the problem is to be solved for a fixed α and y_o with $0 < \alpha < 1$, $y_o \neq 0$ and since the constants $a_1, a_2, u_o, \ldots, u_n$ are arbitrary we can replace a_1 by $\lambda^2 a_1$, etc. and obtain

$$\hat{H}(\omega) = \frac{\lambda^2}{\omega^2+\lambda^2}[a_1 j\omega + a_2\omega^2 + u_o + \sum_{k=1}^{n} u_k e^{-j\omega\tau_k}]P^*(\omega) \tag{113}$$

Since

$$\frac{\lambda^2}{\omega^2+\lambda^2} a_2\omega^2 = a_2\lambda^2 - \frac{a_2\lambda^4}{\omega^2+\lambda^2} \tag{114}$$

so that the Fourier Transform of the optimum filter impulse response can be written as

$$\hat{H}(\omega) = a_2\lambda^2 P^*(\omega) + \frac{\lambda^2}{\omega^2+\lambda^2}[a_1 j\omega + (u_o - a_2\lambda^2) + \sum_{k=1}^{n} u_k e^{-j\omega\tau_k}]P^*(\omega) \tag{115}$$

Since $a_1, a_2, u_o, \ldots, u_n$ arbitrary real constants we can relabel them at our convenience. Similarly the constaint points τ_k can be relabelled appropriately and we can express the optimum solution as

$$\hat{H}(\omega) = a_1 P^*(\omega) + \frac{\lambda^2}{\omega^2+\lambda^2}[a_2 j\omega + \sum_{k=-n}^{n} u_k e^{-j\omega\tau_k}]P^*(\omega) \tag{116}$$

The constants $a_1, a_2, u_{-n}, \ldots, u_n$ are to be chosen so that the solution satisfies the constraint equations (108) - (111).

Up to this point in the analysis, the function $\hat{H}(\omega)$ has been interpreted as the L^2 - Fourier Transform of the optimum filter impulse response. It would

also be convenient if we could interpret $\hat{H}(\omega)$ as the transfer function of the filter. Let us define the functions

$$f(t) = \frac{\lambda}{2} e^{-\lambda |t|} \qquad -\infty < t < \infty \tag{117a}$$

$$g(t) = \sum_{k=-n}^{n} u_k \delta(t-\tau_k) \tag{117b}$$

Note that f(t) can be interpreted as the impulse response of a stable but unrealizable filter, while g(t) is the impulse response of a tapped delay line, ($\delta(t)$ represents the Dirac delta function). Let us also define the function

$$q(t) = a_1 p(t) + f(t) * [a_2 \dot{p}(-t) + g(t) * p(-t)] \tag{118}$$

where "*" denotes the convolution operation. We can take the Fourier Transform of q(t) in the Generalized Function sense to obtain

$$Q(\omega) = a_1 P^*(\omega) + F(\omega)[a_2 j\omega + G(\omega)] P^*(\omega) \tag{119}$$

where $F(\omega)$, $G(\omega)$ are the Fourier Transform of f(t), g(t). In other words

$$F(\omega) = \frac{\lambda^2}{\omega^2 + \lambda^2} \tag{120a}$$

$$G(\omega) = \sum_{k=-n}^{n} u_k e^{-j\omega\tau_k} \tag{120b}$$

But the function $\hat{H}(\omega)$ in Equation (115) is the L^2-Fourier Transform of $\hat{h}(t)$. Therefore it is also the Fourier Transform $\hat{h}(t)$ in the Generalized Function sense, and $\hat{H}(\omega) = Q(\omega)$, in other words,

$$\hat{H}(\omega) = a_1 P^*(\omega) + F(\omega)[a_2 j\omega + G(\omega)] P^*(\omega) \tag{121}$$

From this equation we can draw the block diagram for the filter as shown in Figure 2.*

*Note that in Figure 2, $a_2 = 0$. This corresponds to a result shown in Chapter V.

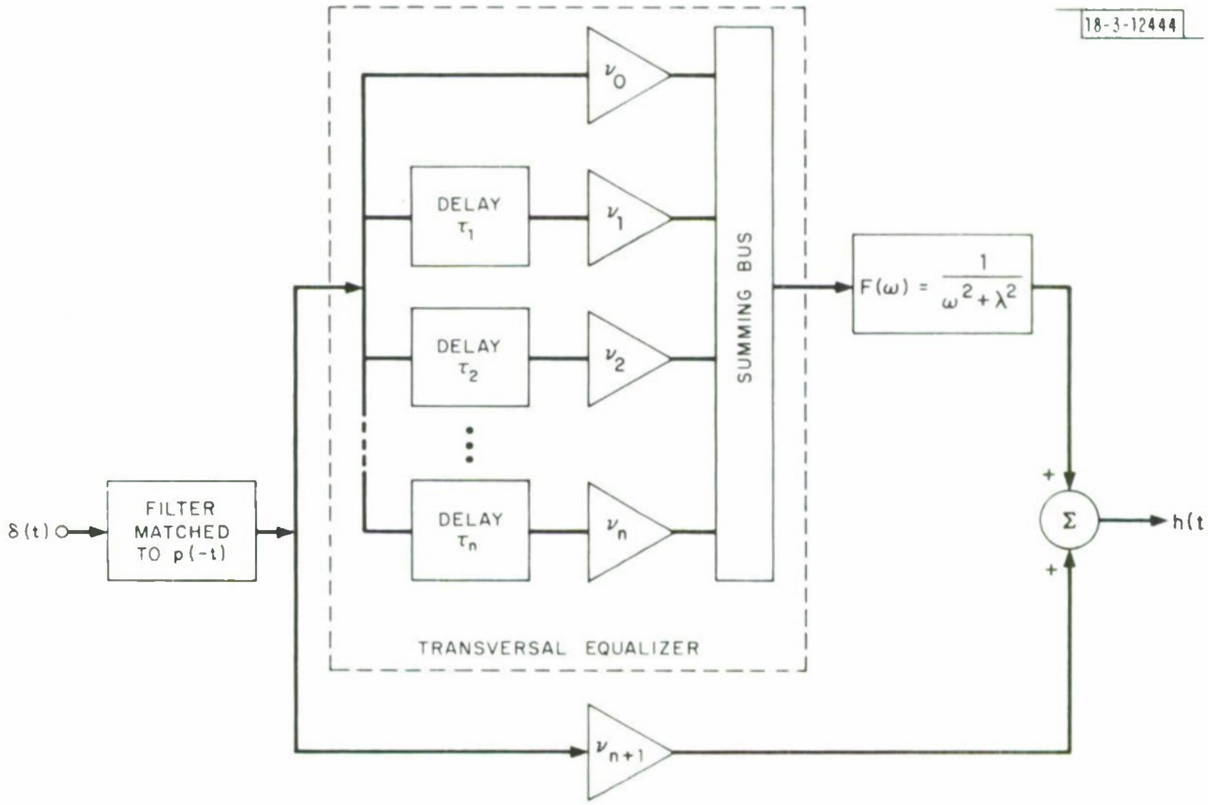


Fig. 2. Block diagram representation of the optimum mismatched filter.

Therefore the optimal processor for estimating the target range and detecting the target in a multiple target environment consists of two parallel processors, one of which is simply a matched filter, the other a matched filter in cascade with a tapped delay line and a bandlimiting filter. If we were to study simply the target detection case in clutter we would have chosen the cost weighting factor, α, in (84) to be zero and we would not have needed the normalization (87) but rather we would set $y_o = 1$. By retracing the steps used to obtain the above solution (this case is even easier) we would have found that the optimum filter was given by (121) with $a_1 = 0$ and $F(\omega) \equiv 1$. Therefore the filter which maximizes the detection signal-to-noise ratio subject to sidelobe constraints is simply a matched filter followed by a tapped delay line. We have therefore established the optimality of the classical transversal equalizer structure.

In other cases, in which $\alpha \neq 0$, the cost function gives some emphasis to the estimation performance of the filter and the optimum structure is that shown in Figure 2. Therefore the transversal equalizer by itself is not the optimum filter, when, in addition to detecting the presence of the target in clutter, its range is to be estimated as well.

In Section VI we have evaluated the performance of the optimum filter for a particular transmitted signal. The dynamics of the various filter parameters are shown at that time. For our purposes at this time, it suffices to say that the solution to the filtering problem has been found using Hilbert Space techniques in the frequency domain. All components in the filter are stable but the band-limiting filter is unrealizable. No questions concerning the duration of the

processing time arise in this case, since the optimum impulse response, $\hat{h}(t) = \mathcal{F}^{-1}[H(\omega)]$, will be defined on the whole real line, $-\infty < t < \infty$. Therefore, the frequency domain formulation does appear to have advantages over the state-space solution. We want to emphasize again, that we found the form of the solution by using the optimal control techniques and used the insights gained from that solution to set up the linear space and the inner product which led to the mathematically rigorous result. Furthermore one advantage of the state-space approach is that it leads directly to a realizable filter.

The problem of filter synthesis is not completely solved, however, since we have yet to determine the constants $a_1, a_2, \mu_{-n}, \ldots, \mu_n$. In the next section we shall show how these parameters can be found by solving a quadratic programming problem.

V. Reduction of the Filter Design to a Nonlinear Programming Problem

In the previous section we showed that for a fixed vector $\underline{y} \in S$, where

$$S = \{ \underline{y}: \ -\epsilon_k y_o \leqq y_k \leqq \epsilon_k y_o, \quad k=\pm 1, \pm 2, \ldots, \pm n\} \tag{122}$$

the function which minimized the combined estimation/detection cost functional

$$\frac{\alpha}{B^2 E} \cdot \int_{-\infty}^{\infty} \omega^2 |H(\omega)|^2 d\omega + \frac{(1-\alpha)}{y_o^2} \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega \tag{123}$$

subject to the constraints

$$\int_{-\infty}^{\infty} j\omega H(\omega) P(\omega) \frac{d\omega}{2\pi} = 0 \tag{124a}$$

$$\int_{-\infty}^{\infty} \omega^2 H(\omega) P(\omega) \frac{d\omega}{2\pi} = B^2 \tag{124b}$$

$$\int_{-\infty}^{\infty} H(\omega) P(\omega) e^{j\omega\tau_k} \frac{d\omega}{2\pi} = y_k \quad k=0, \pm 1, \ldots, \pm n \tag{124c}$$

was of the form

$$\hat{H}(\omega) = a_1 P^*(\omega) + a_2 \frac{j\omega\lambda^2}{\omega^2+\lambda^2} P^*(\omega) + \frac{\lambda^2}{\omega^2+\lambda^2} \left[\sum_{k=-n}^{n} \mu_k e^{-j\omega\tau_k} \right] P^*(\omega) \tag{125}$$

The constants a_1, a_2, $\mu_{-n}, \ldots, \mu_n$ are to be chosen to satisfy constraint equation (124). In this section we propose to show that these constants can be found by solving a nonlinear programming problem. Before we do this, however, we shall show that the $(2n+1)$-dimensional constraint set can be reduced by a

factor of two by assuming that the sidelobe samples are taken symmetrically about $\tau_o = 0$ and that the sidelobe constraint function is symmetric. Only the latter assumption results in some loss in generality of the results. First we define the real-valued functions

$$F_r(\omega) = a_1 + \frac{\lambda^2}{\omega^2+\lambda^2} \sum_{k=-n}^{n} \mu_k \cos\omega\tau_k \tag{126a}$$

$$F_i(\omega) = \frac{a_2\omega\lambda^2}{\omega^2+\lambda^2} - \frac{\lambda^2}{\omega^2+\lambda^2} \sum_{k=-n}^{n} \mu_k \sin\omega\tau_k \tag{126b}$$

and note then that Equation (125) can be written as

$$\hat{H}(\omega) = [F_r(\omega) + jF_i(\omega)]P^*(\omega) \tag{127}$$

Now we make the assumption that the sidelobe sampling points $\{\tau_k\}_{k=-n}^{n}$ be chosen symmetrically, i.e.

$$\tau_k = -\tau_k \tag{128}$$

This assumption results in no loss of generality of the method, but it permits us to rewrite (126) as

$$F_r(\omega) = a_1 + \frac{\lambda^2}{\omega^2+\lambda^2} [\mu_o + \sum_{k=1}^{n} (\mu_k+\mu_{-k})\cos\omega\tau_k] \tag{129a}$$

$$F_i(\omega) = \frac{a_2\omega\lambda^2}{\omega^2+\lambda^2} - \frac{\lambda^2}{\omega^2+\lambda^2} \sum_{k=1}^{n} (\mu_k-\mu_{-k})\sin\omega\tau_k \tag{129b}$$

Let us define new variables according to

$$\upsilon_o = \mu_o,\ \upsilon_{n+1} = a_1,\ \eta_{n+1} = a_2$$

$$\upsilon_k = \mu_k+\mu_{-k},\ \ \eta_k = \mu_k-\mu_{-k} \tag{130}$$

Then the function $F_r(\cdot)$ depends only upon the (n+2)-vector $\underline{\upsilon}$ while $F_i(\cdot)$ depends upon only the (n+1)-vector $\underline{\eta}$. These vectors are to be chosen to satisfy the constraint equation, (124). From (129) it is obvious that

$$F_r(\omega;\underline{\upsilon}) = F_r(-\omega;\underline{\upsilon}) \tag{131a}$$

$$F_i(\omega;\underline{\eta}) = -F_i(-\omega;\underline{\eta}) \tag{131b}$$

where we have explicitly denoted the respective dependence upon $\underline{\upsilon}$ and $\underline{\eta}$. The optimum solution can be written as

$$\hat{H}(\omega;\underline{\upsilon},\underline{\eta}) = [F_r(\omega;\underline{\upsilon})+jF_i(\omega;\underline{\eta})]P^*(\omega) \tag{132}$$

We now substitute for $\hat{H}(\cdot)$ in the constraint equations and obtain the following set of constraint equations which are completely equivalent to those expressed in (124):

$$\int_{-\infty}^{\infty} \omega F_i(\omega;\underline{\eta})|P(\omega)|^2\frac{d\omega}{2\pi} = 0 \tag{133a}$$

$$\int_{-\infty}^{\infty} \omega^2 F_r(\omega;\underline{\upsilon})|P(\omega)|^2\frac{d\omega}{2\pi} = B^2 \tag{133b}$$

$$\int_{-\infty}^{\infty} F_r(\omega;\underline{\upsilon})|P(\omega)|^2\frac{d\omega}{2\pi} = y_o \tag{133c}$$

$$\int_{-\infty}^{\infty} [F_r(\omega;\underline{\upsilon})\cos\omega\tau_k - F_i(\omega;\underline{\eta})\sin\omega\tau_k]|P(\omega)|^2\frac{d\omega}{2\pi} = y_k \tag{133d}$$

These equations follow from the fact that $F_r(\omega;\underline{\upsilon})$ and $|P(\omega)|^2$ are even functions of ω ($\varphi_p(\tau)$ is real) while $F_i(\omega;\underline{\eta})$ is an odd function of ω. For example since $\omega F_r(\omega;\underline{\upsilon})|P(\omega)|^2$ is an odd function of ω

$$\int_{-\infty}^{\infty} \omega F_r(\omega;\underline{\upsilon})|P(\omega)|^2 d\,\omega = 0 \text{ for all } \underline{\upsilon} \tag{134}$$

and the zero-bias constraint is reduced to Equation (133a).

The cost function, (123) depends on the vectors $\underline{\upsilon}$ and $\underline{\eta}$ according to the relation

$$\int_{-\infty}^{\infty} [\frac{\alpha}{B^2E}\omega^2 + \frac{(1-\alpha)}{y_o^2}] [F_r^2(\omega;\underline{\upsilon}) + F_i^2(\omega;\underline{\eta})] |P(\omega)|^2 d\omega \tag{135}$$

In the equations for $F_r(\cdot)$ and $F_i(\cdot)$ we have been using the definition

$$\lambda^2 = \frac{(1-\alpha)}{\alpha} \cdot \frac{B^2E}{y_o^2} \tag{136}$$

These relations have been derived using only the fact that $\tau_{-k} = -\tau_k$. It is now convenient to introduce the assumption that the sidelobe constraint function is symmetric. This condition will be satisfied for the so-called "thumb-tack" constraint function, but there may arise cases where a non-symmetric sidelobe structure would be permissible. We would handle these cases by applying the constraints symmetrically which would implicitly reduce the degree of variation in the multipliers and result in an increased cost. We would like to point out that the assumption is introduced as a matter of convenience in reducing the dimensionality of the problem. We could apply the same analysis to the case of non-symmetric constraint functions, and thereby lose no generality of the method. The symmetry assumption implies that the sidelobe constraint points satisfy the condition

$$y_k = y_{-k} \qquad k=1, 2, \ldots, n \tag{137}$$

Using this condition and the fact that $\tau_{-k} = -\tau_k$, it is easy to show that the side-lobe constraint equation (133d) reduces to requiring only that

$$\int_{-\infty}^{\infty} F_r(\omega;\underline{\upsilon})\cos\omega\tau_k |P(\omega)|^2 \frac{d\omega}{2\pi} = y_k \quad k=1,2,\ldots,n \tag{138}$$

Therefore, for each (n+1)-vector $\underline{y}$ in the set S, where now

$$S = \{\underline{y}: \; -\epsilon_k y_o \leq y_k \leq \epsilon_k y_o, \quad k=1,2,\ldots,n\} \tag{139}$$

we want to pick the constants $\underline{\upsilon}$, $\underline{\eta}$ to satisfy the reduced set of constraint equations, (133a), (133b), (133c), and (139). These are summarized below for convenience.

$$\int_{-\infty}^{\infty} \omega F_i(\omega;\underline{\eta})|P(\omega|^2 \frac{d\omega}{2\pi} = 0 \tag{140a}$$

$$\int_{-\infty}^{\infty} \omega^2 F_r(\omega;\underline{\upsilon})| P(\omega)|^2 \frac{d\omega}{2\pi} = B^2 \tag{140b}$$

$$\int_{-\infty}^{\infty} F_r(\omega;\underline{\upsilon})| P(\omega)|^2 \frac{d\omega}{2\pi} = y_o \tag{140c}$$

$$\int_{-\infty}^{\infty} F_r(\omega;\underline{\upsilon})\cos\omega\tau_k | P(\omega)|^2 \frac{d\omega}{2\pi} = y_k, \quad k=1,2,\ldots,n \tag{140d}$$

and the associated value of the cost functional is, from Equation (135)

$$\frac{\alpha}{B^2 E} \int_{-\infty}^{\infty} (\omega^2+\lambda^2)[F_r^{\,2}(\omega;\underline{\upsilon})+F_i^{\,2}(\omega;\underline{\eta})]| P(\omega)|^2 d\omega \tag{141}$$

Notice, that for each fixed value of $\underline{y}\in S$, that only the zero-bias constraint, Equation (140a), depends upon the vector $\underline{\eta}$. All of the other constraints are satisfied by the appropriate choice of $\underline{\upsilon}$, independent of $\underline{\eta}$. At the same time, the cost function is increased for every non-zero value of $\underline{\eta}$. From (129b) and (130),

$$F_i(\omega;\underline{\eta}) = \frac{\omega\lambda^2}{\omega^2+\lambda^2}\eta_{n+1} - \frac{\lambda^2}{\omega^2+\lambda^2}\sum_{k=1}^{n}\eta_k \sin\omega\tau_k \tag{142}$$

from which it is clear that zero-bias constraint can be satisfied and the cost functional minimized by choosing $\underline{\eta} = \underline{0}$, i.e. $F_i(\omega;\underline{\eta}) = 0$. Relating this condition to the definition, (130), we see that this choice of $\underline{\eta}$ is indeed possible by picking

$$a_2 = 0, \quad \mu_k = \mu_{-k} \quad k=1,2,\ldots,n \tag{143}$$

Therefore, the symmetry of the sidelobe constraint function permits us to reduce the dimensionality of the constraint space by a factor of 2 and we no longer need concern ourselves with the zero-bias constraint since this will be satisfied automatically by the above choice of multipliers.

Summarizing these results, we conclude that when the sidelobe constraint function is symmetric, the optimum solution to the filtering problem is

$$\hat{H}(\omega;\underline{\upsilon}) = F(\omega;\underline{\upsilon})P^*(\omega) \tag{144}$$

where

$$F(\omega;\underline{\upsilon}) = \frac{\lambda^2}{\omega^2+\lambda^2}\sum_{k=0}^{n}\upsilon_k\cos\omega\tau_k + \upsilon_{n+1} \tag{145}$$

and the performance of this filter is measured by the cost functional

$$\frac{\alpha}{B^2E}\int_{-\infty}^{\infty}(\omega^2+\lambda^2)F^2(\omega;\underline{\upsilon})|P(\omega)|^2 d\omega \tag{146}$$

where

$$\lambda^2 = \frac{(1-\alpha)}{\alpha}\cdot\frac{B^2E}{y_o^2} \tag{147}$$

and where the (n+2)-vector $\underline{\upsilon}$ is chosen to satisfy the constraints

$$\int_{-\infty}^{\infty} \omega^2 F(\omega;\underline{\upsilon}) |P(\omega)|^2 \frac{d\omega}{2\pi} = B^2 \tag{148a}$$

$$\int_{-\infty}^{\infty} F(\omega;\underline{\upsilon}) |P(\omega)|^2 \frac{d\omega}{2\pi} = y_o \tag{148b}$$

$$\int_{-\infty}^{\infty} F(\omega;\underline{\upsilon}) \cos\omega\tau_k |P(\omega)|^2 \frac{d\omega}{2\pi} = y_k \quad k=1, 2, \ldots, n \tag{148c}$$

In a crude sense, we could imagine picking a vector $\underline{y}$ in the set S, solving the preceeding (n+2) constraint equations for the (n+2) vector $\underline{\upsilon}$ which can then be used to obtain a value of the cost function for that particular $\underline{y}$-vector. This could be done for all possible $\underline{y}$-vectors in S and the particular $\underline{y}$ which minimized the cost could be found. In the next few paragraphs we shall show how this search procedure can be performed in a systematic way using quadratic programming techniques. We begin the analysis by first manipulating the constraint equation (148). From (147) we notice that for each fixed value of y_o, λ^2 is a fixed positive number. Therefore, if Equation (148b) holds, the following equation holds,

$$\int_{-\infty}^{\infty} \lambda^2 F(\omega;\underline{\upsilon}) |P(\omega)|^2 \frac{d\omega}{2\pi} = \frac{(1-\alpha)}{\alpha} \cdot \frac{B^2 E}{y_o} \tag{149}$$

Then if (148b) holds, constraint (148a) is equivalent to requiring that

$$\int_{-\infty}^{\infty} (\omega^2+\lambda^2) F(\omega;\underline{\upsilon}) |P(\omega)|^2 \frac{d\omega}{2\pi} = B^2 \left[1+ \frac{(1-\alpha)}{\alpha} \frac{E}{y_o}\right] \tag{150}$$

Therefore for each $\underline{y} \in S$ we solve for the vector $\underline{\upsilon}$ which satisfies the constraints, Equations (148b), (148c), and (150), namely

$$\int_{-\infty}^{\infty} F(\omega;\underline{\upsilon}) |P(\omega)|^2 \frac{d\omega}{2\pi} = y_o \tag{151a}$$

$$\int_{-\infty}^{\infty} F(\omega;\underline{\upsilon})\cos\omega\tau_k \left| P(\omega)\right|^2 \frac{d\omega}{2\pi} = y_k \qquad k=1, 2, \ldots, n \tag{151b}$$

$$\int_{-\infty}^{\infty} (\omega^2+\lambda^2)F\ (\omega;\underline{\upsilon}) \left| P(\omega)\right|^2 \frac{d\omega}{2\pi} = B^2[1+\frac{(1-\alpha)}{\alpha}\frac{E}{y_o}] \tag{151c}$$

and the associated cost will be

$$\frac{\alpha}{B^2 E}\int_{-\infty}^{\infty} (\omega^2+\lambda^2)F^2(\omega;\underline{\upsilon}) \left| P(\omega)\right|^2 d\omega \tag{152}$$

where

$$F(\omega;\underline{\upsilon}) = \frac{\lambda^2}{\omega^2+\lambda^2}\sum_{k=0}^{n} \upsilon_k \cos\omega\tau_k + \upsilon_{n+1} \tag{153}$$

Now define the complex vector-valued function $\underline{V}(\omega)$ as

$$\underline{V}(\omega) = \frac{P(\omega).\lambda^2}{-j\omega+\lambda}\begin{bmatrix} 1 \\ \cos\omega\tau_1 \\ \cos\omega\tau_2 \\ \cdot \\ \cdot \\ \cdot \\ \cos\omega\tau_n \\ \frac{\omega^2+\lambda^2}{\lambda^2} \end{bmatrix} \tag{154}$$

and the (n+2) x (n+2) non-negative definite matrix

$$\Gamma = \frac{1}{\lambda^2}\int_{-\infty}^{\infty} \underline{V}^*(\omega)\underline{V}'(\omega)\frac{d\omega}{2\pi} \tag{155}$$

Here the asterisk denotes complex conjugate while the prime denotes vector transpose. It is easy to show by straight forward manipulations that the constraint equations, Equation (151) can be written more compactly as

$$\Gamma\underline{\upsilon} = \begin{bmatrix} y_o \\ y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \\ \frac{B^2}{\lambda^2}\left[1+\frac{(1-\alpha)}{\alpha}\cdot\frac{E}{y_o}\right] \end{bmatrix} \tag{156}$$

and for each value of $\underline{\upsilon}$ satisfying these constraints the associated cost is

$$f(\underline{\upsilon}) = \frac{\alpha\lambda^2}{B^2 E}(\underline{\upsilon}'\Gamma\underline{\upsilon}) \tag{157}$$

Since the vector $\underline{y}\in S$, then

$$-\epsilon_k y_o \leqq y_k \leqq \epsilon_k y_o \quad k=1, 2, \ldots, n \tag{158}$$

Because the constraints on $\underline{\upsilon}$ are actually these inequality constraints there is a certain amount of freedom in the selection of the $\underline{\upsilon}$-vector. We would like to find the particular value of $\underline{\upsilon}$ within prescribed class which minimizes the cost function. Notice that for each value of y_o, λ^2 is fixed number and therefore the vector $\underline{V}(\omega)$ is a completely known function. Consequently the matrix Γ is fixed also, but it changes as y_o changes. Let us denote this dependence explicitly by writing $\underline{V}(\omega;y_o)$ and $\Gamma(y_o)$. Then for each value of y_o we want to find the vector $\underline{\upsilon}$ which minimizes the cost function

$$f(\underline{\upsilon};y_o) = \frac{\alpha\lambda^2(y_o)}{B^2 E}\cdot(\underline{\upsilon}'\Gamma(y_o)\underline{\upsilon}) \tag{159}$$

subject to the constraints

$$
\begin{bmatrix} y_o \\ -\epsilon_1 y_o \\ -\epsilon_2 y_o \\ \cdot \\ \cdot \\ \cdot \\ -\epsilon_n y_o \\ \frac{B^2}{\lambda^2(y_o)}[1+\frac{(1-\alpha)}{\alpha}\frac{E}{y_o}] \end{bmatrix} \leqq \Gamma(y_o)\underline{\upsilon} \leqq \begin{bmatrix} y_o \\ \epsilon_1 y_o \\ \epsilon_2 y_o \\ \cdot \\ \cdot \\ \cdot \\ \epsilon_n y_o \\ \frac{B^2}{\lambda^2(y_o)}[1+\frac{(1-\alpha)}{\alpha}\frac{\mathbf{E}}{y_o}] \end{bmatrix} \tag{160}
$$

which is a nonlinear programming problem. Since (160) represents linear inequality constraints, Hildreth's Method [9] can be used to transform this optimization problem to a quadratic programming problem of the form: Find the vector $\underline{x} \geqq \underline{0}$ which minimizes the cost function $\underline{c}'\underline{x} + \underline{x}'D\underline{x}$. Efficient computational algorithms which guarantee convergence to an optimum $\underline{x}$ in a finite number of steps are available to solve this latter problem [10].

Therefore for each y_o, we can solve for the minimizing $\underline{\upsilon} = \underline{\upsilon}(y_o)$ which has associated with it the cost $f(\underline{\upsilon}(y_o);y_o)$. We then use conventional search techniques [11] to search over all possible values of y_o to find the pair $(\underline{\upsilon}^*, y_o^*)$ where $\underline{\upsilon}^* = \underline{\upsilon}(y_o^*)$ which minimized $f(\underline{\upsilon}(y_o), y_o)$ treated as a function of y_o. This technique can be implimented on the computer in a relatively straightforward way. The major difficulty lies in the evaluation of the matrix $\Gamma(y_o)$ defined in Equations (154) and (155). For reasonable choices of the transmitted signal, the evaluation can often times be performed analytically without recourse to the digital computer.

Therefore the optimum set of multipliers $\{\upsilon_k\}_{k=0}^{n+1}$ and the filter pole location, described by

$$\lambda^2 = \frac{(1-\alpha)}{\alpha} \cdot \frac{B^2 E}{y_o^2} \tag{161}$$

can be found for any weighting factor α. The resulting optimum filter is

$$\hat{H}(\omega) = F(\omega;\underline{\upsilon})P*(\omega) \tag{162}$$

where

$$F(\omega;\underline{\upsilon}) = \frac{\lambda^2}{\omega^2+\lambda^2} \sum_{k=0}^{n} \upsilon_k \cos\omega\tau_k + \upsilon_{n+1} \tag{163}$$

The actual estimate variance and detection signal-to-noise ratio can then be evaluated for any value of the weighting parameter α. We would construct a table of such values. Then for a prescribed application the desired detection signal-to-noise performance would be specified and we would reference the table we had constructed to find the appropriate value of α. With this weighting we could then solve for the correct values of the multipliers and thereby generate the filter which gives the desired detection performance and yields the smallest estimation variance. In the next section we shall apply this methodology to a practical design problem.

VI. Numerical Results

In order to perform a numerical study of the dynamics of the parameters involved in the optimum mismatched filter, it is necessary to evaluate the matrix

$$\Gamma(y_o) = \frac{1}{\lambda^2} \int_{-\infty}^{\infty} \underline{V}^*(\omega;y_o)\underline{V}'(\omega;y_o)\frac{d\omega}{2\pi} \tag{164}$$

where

$$\underline{V}(\omega;y_o) = \frac{P(\omega)\lambda^2}{-j\omega+\lambda} \begin{bmatrix} 1 \\ \cos\omega\tau_1 \\ \cos\omega\tau_2 \\ \cdot \\ \cdot \\ \cdot \\ \cos\omega\tau_n \\ \frac{\omega^2+\lambda^2}{\lambda^2} \end{bmatrix} \tag{165}$$

$$\lambda^2 = \frac{(1-\alpha)}{\alpha} \cdot \frac{B^2E}{y_o^2} \tag{166}$$

Conceptually this evaluation can always be done since once the transmitted signal is specified, $P(\omega)$ could be found and the integrations performed. This could be a costly exercise if recourse to a digital computer need be made, since the matrix is of large dimension and would have to be evaluated for possibly many values of y_o. For our purposes, we mainly want to demonstrate the feasibility of the design method and we shall therefore restrict our interest to a class of signals for which the matrix computation can be done analytically. We motivate our choice by noting that the vector $\underline{V}(\omega;y_o)$ could be written more generally as

$$\underline{V}(s;y_o) = \frac{P(s)\lambda^2}{-s+\lambda} \begin{bmatrix} 1 \\ \frac{1}{2}e^{s\tau_1} + \frac{1}{2}e^{-s\tau_1} \\ \frac{1}{2}e^{s\tau_2} + \frac{1}{2}e^{-s\tau_2} \\ \vdots \\ \frac{1}{2}e^{s\tau_n} + \frac{1}{2}e^{-s\tau_n} \\ \frac{(-s+\lambda)(s+\lambda)}{\lambda^2} \end{bmatrix} \tag{167}$$

where we have merely replaced $j\omega$ by the complex variable $s = \sigma + j\omega$. Then the integral, Equation (164) can be viewed as the value of the contour integral of the complex function

$$\Gamma(y_o) = \frac{1}{\lambda^2} \int_{-j\infty}^{j\infty} \underline{V}(-s;y_o)\underline{V}'(s;y_o)\frac{ds}{2\pi j} \tag{168}$$

where the contour is chosen in the s-plane along the $j\omega$ axis. Then if we choose our time functions so that

$$P(s) = \sum_{m=1}^{m} \frac{a_m}{(s+s_m)(s+s_m^*)} \tag{169}$$

then we can actually evaluate the integral using the residue theorem. Our results are further restricted to the particularly simple signal for which

$$P(s) = \frac{a}{(s+s_p)(s+s_p^*)} \tag{170}$$

We shall choose the constant, a, to normalize the transmitted signal energy, i.e.

$$E = \int_{-\infty}^{\infty} |P(j\omega)|^2 \frac{d\omega}{2\pi} = 1 \tag{171}$$

and we can then compute

$$B^2 = \int_{-\infty}^{\infty} \omega^2 |P(j\omega)|^2 \frac{d\omega}{2\pi} \tag{172}$$

which now represents the second moment bandwidth of the signal. We can therefore change the signal bandwidth by varying the pole location s_p. We then used (170) in (168) and applied the residue theorem as proposed. This left us with a closed form expression for $\Gamma(y_o)$ in terms of y_o, and this was easily programmed on the digital computer.

If we represent s_p in (170) by $\sigma_p + j\omega_p$, the compressed pulse for a matched filter is given by

$$\varphi(t) = e^{-\sigma_p t}(\cos\omega_p t + \frac{\sigma_p}{\omega_p} \sin \omega_p t), \ t \geq 0 \tag{173}$$

with $\varphi(-t) = \varphi(t)$. The mean-square bandwidth of the signal is easily calculated to be

$$B^2 = \omega_p^2 + \sigma_p^2 \tag{174}$$

By differentiating (173), the subsidiary peaks of the compressed pulse are found to be located at $t_k = k\pi/\omega_p$, $|k| = 1, 2, \ldots$, with the peak magnitudes given by $|\varphi(t_k)| = \exp(-|k| \frac{\pi\sigma_p}{\omega_p})$. The choice of $\omega_p = 4\pi\sigma_p$ results in four sidelobes within a single time-constant ($\triangleq \sigma_p^{-1}$), the first peak magnitude being -2.3Db with respect to the main peak. The compressed pulse and its spectrum for a signal $\sigma_p = 1$, $\omega_p = 4\pi$ are shown in Figure 3.

To illustrate the techniques developed in this paper, constraint levels corresponding to -10 Db and -20 Db sidelobes were considered. In each case, the constraint was applied at 51 evenly spaced points in the closed interval [0.25, 2.0].

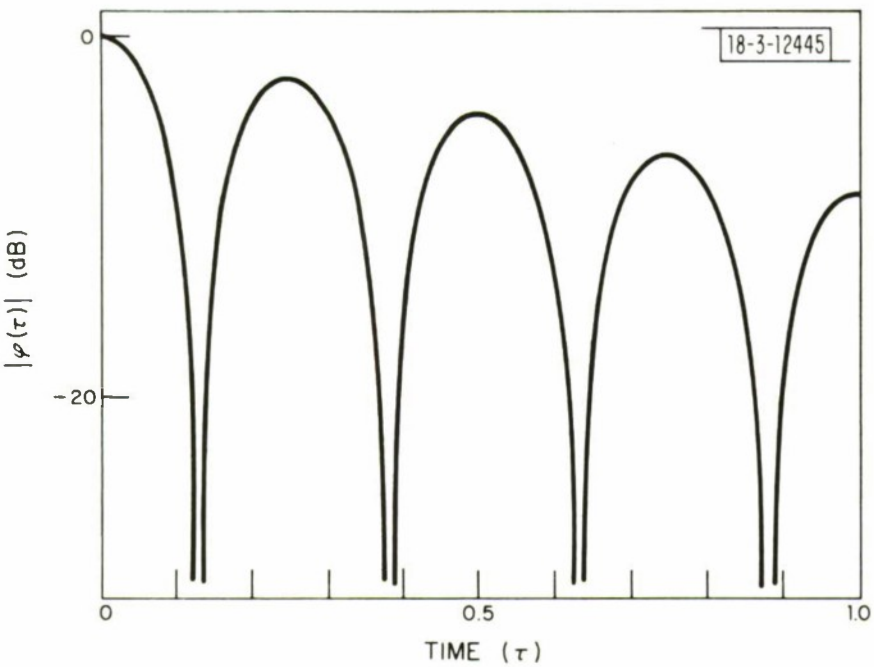


Fig. 3a. Matched filter compressed pulse.

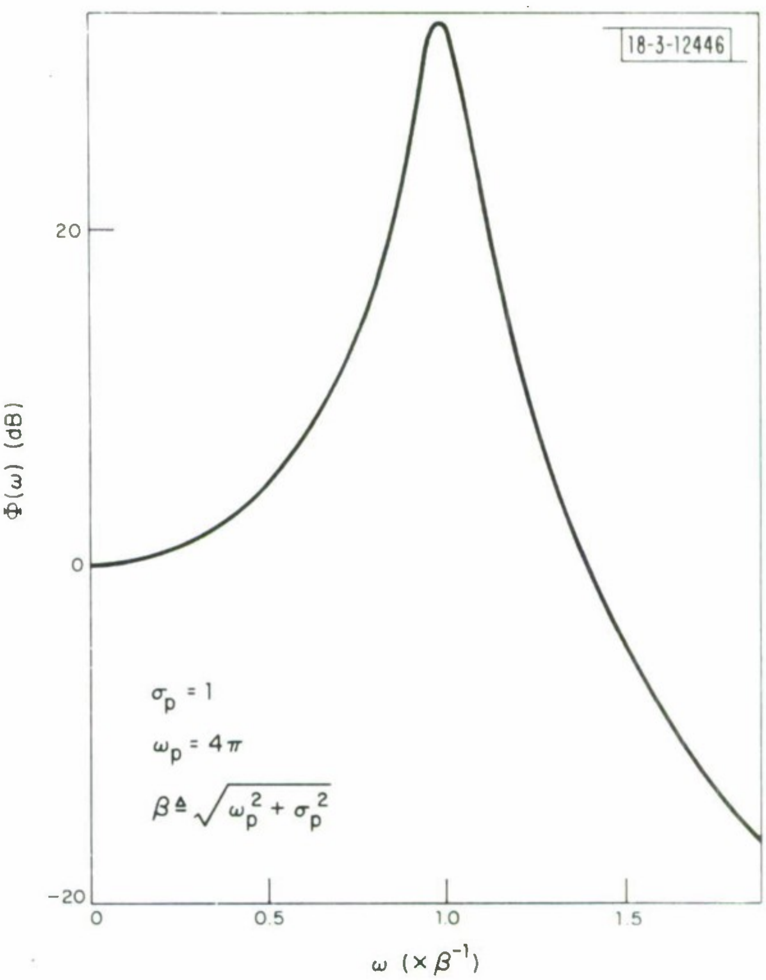


Fig. 3b. Compressed pulse spectrum.

Relating this problem to the development in Section 5, we have chosen $\tau_n = \frac{n-1}{N-1}(\tau_N - \tau_1) + \tau_1$, $\epsilon_n = \epsilon$, $n=1, \ldots 51=N$ where $\tau_1 = 0.25$ and $\tau_N = 2.0$. For a fixed α-value, the quadratic programming problem determined by (159) and (160) is solved for several values of y_o. The minimum cost function is plotted vs y_o in Figure 4 for $\alpha=0.5$. For any arbitrary α, the minimizing value of y_o can be determined by such graphical means and will be denoted by $y_o^*(\alpha)$. The solution of the quadratic programming problem specified by α, $y_o^*(\alpha)$ will be denoted by $\underline{\upsilon}^*(\alpha)$. The bandwidth parameter for this optimum solution can be calculated from (161) with $y_o = y_o^*(\alpha)$ and will be denoted by $\lambda^*(\alpha)$

Recall that the cost function given by (159) is a weighted sum of the estimation and detection performance terms. We wish to determine the performance values separately for the optimum processors $H^*(\omega;\alpha)$ which are obtained from (162) and (163) with $\underline{\upsilon} = \underline{\upsilon}^*(\alpha)$ and $\lambda = \lambda^*(\alpha)$. The resulting estimation performance, $\mathcal{E} \triangleq \frac{\sigma^2_{mmf}}{\sigma^2_{mf}}$, is obtained as a function of α and is calculated by the formula

$$\mathcal{E}(\alpha) = B^{-2} \int_{-\infty}^{+\infty} \omega^2 [F^*(\omega;\alpha)]^2 \,|P(\omega)|^2 \frac{d\omega}{2\pi} \tag{175}$$

where

$$F^*(\omega;\alpha) \triangleq \frac{1}{\left(\frac{\omega}{\lambda^*(\alpha)}\right)^2 + 1} \sum_{k=o}^{N} \upsilon_k^* \cos\omega\tau_k + \upsilon_{N+1}^* \tag{176}$$

The detection performance, $\mathcal{D} \triangleq \frac{\rho_{mmf}}{\rho_{mf}}$, can be calculated by

$$\mathcal{D}(\alpha) = \frac{E}{[y_o^*(\alpha)]^2} \int_{-\infty}^{+\infty} [F^*(\omega)]^2 \frac{d\omega}{2\pi} \tag{177}$$

From (175) and (177), the optimum trade-off between detection and estimation performance can be determined by constructing the graph given by the set of ordered pairs $\{(\mathcal{D}^{-1}(\alpha), \mathcal{E}^{-1}(\alpha)): 0<\alpha<1\}$. For the particular signal chosen, two such graphs corresponding to two sidelobe constraint levels are shown in Figure 5. For any given level of detection performance ($\mathcal{D}^{-1}$ corresponds to effective SNR loss due to mismatching), the graphs of Figure 5 indicate the best possible range estimation which can be attained subject to the appropriate sidelobe constraint (and, of course, the original assumptions on receiver structure).

For purposes of comparison, the performance of a well-known sub-optimal processor using the clutter-rejection filter was calculated. The clutter rejection filter is specified by [12], [13]

$$H_{cr}(\omega) = \frac{P^*(\omega)}{\rho + |P(\omega)|^2} \tag{178}$$

The sidelobe level achieved by this filter depends on ρ. For our particular choice of signal, the compressed pulse and the performance of H_{cr} can be analytically determined as a function of ρ using standard LaPlace transform techniques. The analysis is somewhat lengthy, however, so it is omitted. The performance of the clutter rejection filter for several sidelobe levels is indicated on Figure 5 for comparison with the optimal values.

A typical compressed pulse and its spectrum resulting from the optimum filter for $\alpha = 0.5$ is shown in Figure 6. The sidelobe constraint level was -20Db, corresponding to a peak sidelobe magnitude of 0.1 . The sidelobe structure of the compressed pulse actually exceeds this level by a slight

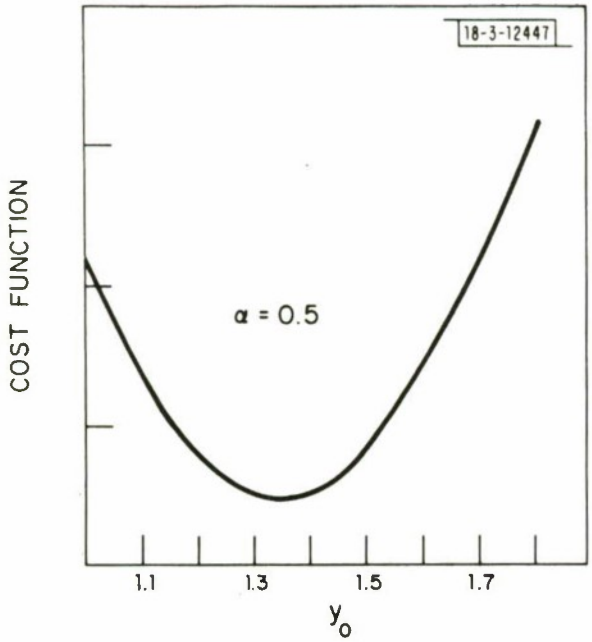


Fig. 4. Cost function vs y_o.

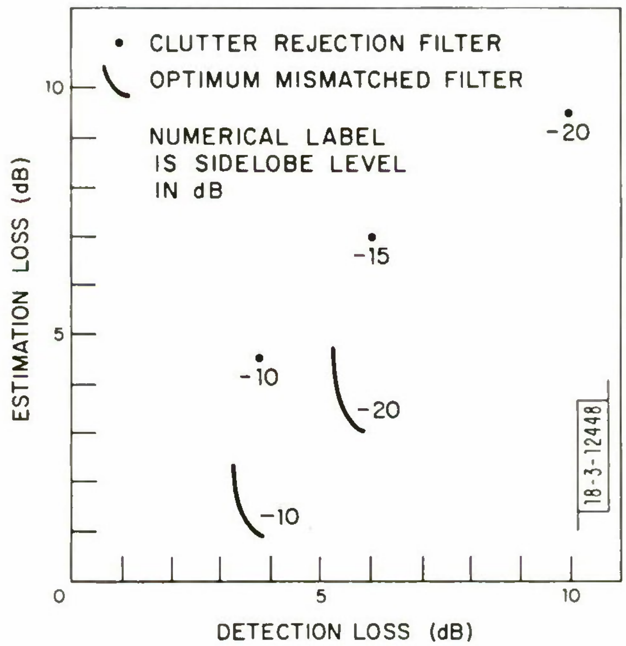
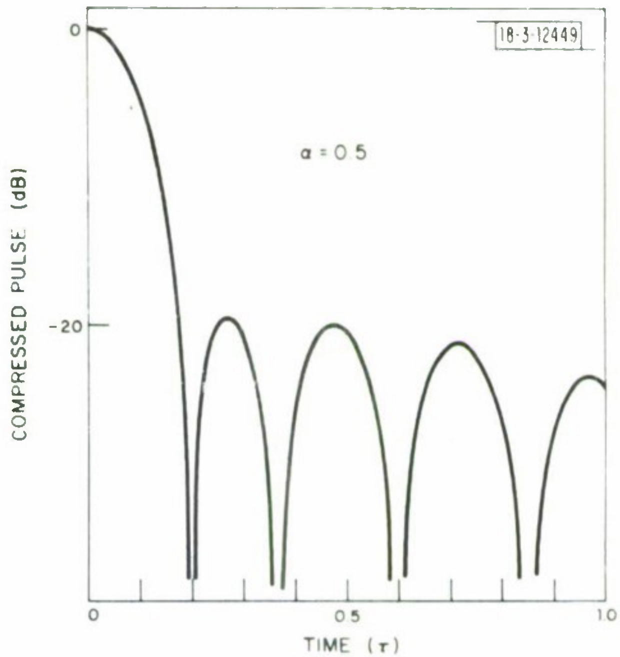


Fig. 5. Detection and estimation performance of optimum mismatched filter.

Fig. 6a. Optimum mismatched filter compressed pulse.



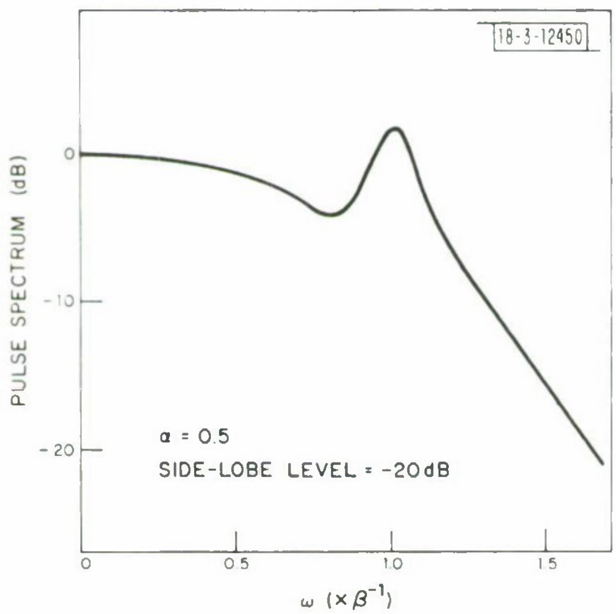


Fig. 6b. Compressed pulse spectrum.

amount. This is due to the fact that the constraint is imposed at only a finite number of points. In between these constraint points, the sidelobe level may exceed the desired value. This effect, if observed, may be reduced by simply taking more constraint points in the violated region. This procedure does not necessarily increase the number of required taps in the delay line realization of the optimum filter as may appear to be the case. A property of quadratic programming problems, when interpreted in the context at hand, states that the gain of a particular tap will be zero unless the constraint corresponding to that tap (Equation (160)) is satisfied with equality. In other words, if υ_n^*, the gain of the tap corresponding to a τ_n second delay (advance) is non-zero, then the value of the compressed pulse at τ_n seconds must be either $\pm \epsilon_n y_o^*$. In most cases, the number of times this condition occurs is determined more by the number of subsidiary peaks in the compressed pulse structure than by the number of constraint points chosen. For the case studied here, the number of non-zero tap gains (single-sided) varied from one to four.

VII. Conclusions

The results of the preceding section show that the techniques developed in this paper can produce a significant level of sidelobe reduction while minimizing loss in effective signal-to-noise ratio and range estimation accuracy. Specifically, we have considered the problem of minimizing a weighted sum of the estimation and detection performance over a broad class of receivers subject to a sidelobe constraint. We have obtained the solution to this problem for an extremely large class of signals, specifically those with finite energy and finite mean-square bandwidth and with a symmetric spectrum; furthermore, the structure of the optimum filter is quite simple.

For the case in which detection performance alone is considered, the optimal filter consists of a matched filter followed by a transversal filter, the so-called transversal equalizer. We have therefore established the optimality of this well-known receiver structure for the case of target detection. In the more general case where estimation accuracy is considered, the transversal equalizer is no longer the optimum processor; further filtering is required. The nature of this additional filtering is such that the width of the main peak of the compressed pulse is increased as emphasis is shifted to estimation performance, a somewhat surprising result since widening of the central peak is often considered synononomous with a reduction in estimation accuracy. In Figure 7, the compressed pulses of the optimum filters for three different values of α are shown. As the estimation performance is emphasized (α is increased), a widening of the pulse is observed. In order

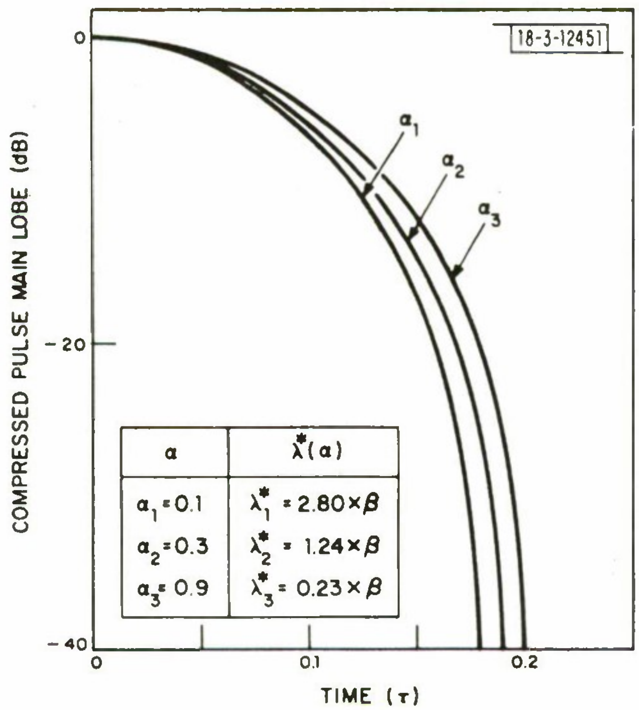


Fig. 7. Comparison of compressed pulse main lobe.

to understand this phenomenon, the basic equation for the normalized estimation variance, $\frac{\sigma^2_{mmf}}{\sigma^2_{mf}}$ is written in the following form

$$\left(\overline{\omega^2}_{mmf} T^2_{mmf}\right) \cdot \left(T^2_{mmf}/T^2_{mf}\right) \cdot \left(\frac{\rho_{mmf}}{\rho_{mf}}\right)$$

$T_{mmf} \stackrel{\Delta}{=} 1/\sqrt{-\ddot{y}(0)/y(0)}$ where y(t) is the compressed pulsed of a mismatched filter, h(t). $\overline{\omega^2}_{mmf}$ is the mean-square bandwidth of h(t). Note that T_{mmf} is a measure of the width of the compressed pulse. For a matched filter this pulsewidth is written as T_{mf} and is given by

$T_{mf} = 1/\sqrt{\overline{\omega^2}_{mf}}$ where $\overline{\omega^2}_{mf}$ is the mean-square bandwidth of the signal (and the matched filter). As the weighting is shifted from detection to estimation performance, the detection mismatch loss, $\frac{\rho_{mmf}}{\rho_{mf}}$, naturally increases. Since the pulse width ratio, T_{mmf}/T_{mf} also increases with α, as indicated by Figure 7, the only way the estimate variance can possibly decrease is for the filter bandwidth, $\overline{\omega^2}_{mmf}$, to be reduced. The necessary bandwidth reduction is accomplished by decreasing the parameter λ^*. The table in Figure 7 shows λ^* for the three pertinent values of α. The characteristic of the optimum filter, at least for the signal considered, we note, is that good estimation performance is achieved by allowing the compressed pulse to spread by simultaneously reducing filter bandwidth. In sharp contrast, the clutter-rejection filter behaves in the opposite manner. The compressed pulse of this filter is actually narrower than the matched filter output, but the bandwidth is substantially greater, [13].

The most restrictive assumption made in this paper was that the compressed pulse (complex representation) be real. This restriction excludes the possibility of a non-symmetric signal spectrum. Also, the possibility of treating perturbations caused by an imperfectly constructed matched filter or by a distorted input signal is denied since both these situations lead to a complex compressed pulse in general. If the compressed pulse is allowed to be complex, the performance criteria (estimate variance and detection mismatch loss) must be modified to include a frequency bias term. The more general formulas may be found in [3]. Furthermore, the sidelobe constraints will be quadratic in nature rather than linear. An equivalent problem for this more general case would probably have to involve a more general convex programming algorithm and an additional search over the frequency bias term.

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REFERENCES

[1] McAulay, R.J., "Numerical Optimization Techniques Applied to PPM Signal Design," IEEE Transactions on Information Theory, Vol. IT-14, No. 5, pp. 708-716, (September 1968).

[2] Cook, C.E., Bernfeld.M., Radar Signals. Academic Press, Chapter 7 New York, (1967).

[3] Hansen, V.G., "The Accuracy of Radar Range Measurements Using Non-Optimum Filters," IEEE Transactions on Aerospace and Navigational Electronics, Vol. ANE-12, pp. 134-140, (June 1965).

[4] Kelly, E.J., Reed, I.S., Root, W.L., "The Detection of Radar Echoes in Noise " Part II, J.Soc. Indust. Appl. Math. Vol. 8, No. 3, pp. 481-507, (September 1960).

[5] Fortmann, T.E., "Optimum Design of Signals and Filters Subject to Sidelobe Constraints," Electronics Systems Laboratory Report ESL-R-400, Massachusetts Institute of Technology, (September 1969).

[6] Athans, M., Falb, P.L., Optimal Control, McGraw-Hill, New York, pp. 498-500, (1966).

[7] Ibid, pp. 284-308.

[8] Ibid, pp. 481-496.

[9] Hadley, G., Nonlinear and Dynamic Programming, Addison-Wesley, Reading, Massachusetts, pp. 234-235, (1964).

[10] Ibid, Chapter 7.

[11] Wilde, D.J., Optimum Seeking Methods, Prentice Hall, Englewood Cliffs, New Jersey, (1964).

[12] Cook and Bernfeld, op. cit., pp. 218-219.

[13] Johnson, J.R., "The Clutter Rejection Filter" (to be published).

[14] McAulay, R.J., Johnson, J.R., "Optimal Mismatched Filter Design for Radar Ranging and Resolution," Technical Note 1969-6, Lincoln Laboratory, M.I.T. (8 May 1969), DDC AD-690994.

[15] Johnson, J.R., "Optimum Detection and Range Estimation Subject to Ambiguity Constraints" (to be published).

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13. ABSTRACT

In a multiple target environment a radar signal processor often uses weighting filters which are not necessarily matched to the transmitted waveform. In this paper expressions for the mean-square range-estimation error, the detection signal-to-noise (SNR) and the effects of sidelobes are derived in terms of the impulse response of an arbitrary mismatched filter. It is desired to find that impulse response which results in the minimum range estimate variance subject to preassigned constraints on the sidelobes and the detection SNR. This optimization problem is first formulated in state-space in which the optimal control law is sought. Pontryagin's maximum principle is used to obtain necessary conditions for the optimum impulse response, from which it is possible to deduce the structure of the optimum filter. Certain mathematical details which detract from the rigor of the time domain formulation are resolved by formulating the problem in the frequency domain and applying Hilbert Space techniques. It is shown that for the problem of detecting the radar target and estimating its range, the optimum filter is a modified transversal equalizer. If only the detection function is to be performed the optimum filter reduces to the transversal equalizer. This establishes the optimality of this important practical device as the solution to the radar detection problem in a multiple target environment. The tap weights and spaces of the delay line as well as certain other parameters upon which the solution depends can be found by solving a non-linear programming problem. Numerical results are given for an interesting class of transmitted waveforms which shows the tradeoffs of the various filter parameters.

14. KEY WORDS

mismatched filters
radar ranges
detection
resolution
sidelobe constraints
optimal control
Hilbert Space
nonlinear programming